Mathematical methods for machine learning and signal processing SS 19

Solutions to problem set 2

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Problem 1: Eigenvalue decomposition

1. Any hermitian matrix $\mathbf{T}$ can be written as $\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{\mathsf{H}}$, where $\mathbf{U}$ is unitary ($\mathbf{U}^{\mathsf{H}} = \mathbf{U}^{-1}$)

$$\begin{aligned}
\mathbf{T}(\mathbf{I}+\mathbf{T})^{-1} &= \mathbf{U}\Lambda\mathbf{U}^{\mathsf{H}}(\mathbf{I}+\mathbf{U}\Lambda\mathbf{U}^{\mathsf{H}})^{-1} \\
&= \mathbf{U}\Lambda\mathbf{U}^{\mathsf{H}}(\mathbf{U}(\mathbf{U}^{\mathsf{H}}\mathbf{I}\mathbf{U}+\Lambda)\mathbf{U}^{\mathsf{H}})^{-1} \\
&= \mathbf{U}\Lambda\mathbf{U}^{\mathsf{H}}(\mathbf{U}^{\mathsf{H}})^{-1}(\mathbf{I}+\Lambda)^{-1}\mathbf{U}^{-1} \\
&= \mathbf{U}\Lambda(\mathbf{I}+\Lambda)^{-1}\mathbf{U}^{-1} \\
&= \mathbf{U}\begin{bmatrix} \frac{\lambda_1}{(1+\lambda_1)} & & \\ & \ddots & \\ & & \frac{\lambda_n}{(1+\lambda_n)} \end{bmatrix}\mathbf{U}^{\mathsf{H}}.
\end{aligned}$$

We see that $\mathbf{T}(\mathbf{I}+\mathbf{T})^{-1}$ has the same eigenvectors as $\mathbf{T}$ and eigenvalues $\tilde{\lambda}_i = \frac{\lambda_i}{1+\lambda_i}$.

2. In general it is not possible to say anything about the eigenvalues of a product of arbitrary $N \times N$ hermitian matrixes with known eigenvalues. To see this consider the two sets of matrices

$$\{\mathbf{A}, \mathbf{B}_1\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right\}$$

and

$$\{\mathbf{A}, \mathbf{B}_2\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ have the same eigenvalues. However, the products

$$\mathbf{A}\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$\mathbf{A}\mathbf{B}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

have different eigenvalues.

3. The fact that $\mathbf{S}$ and $\mathbf{T}$ have the same eigenvectors, implies that there is a unitary matrix $\mathbf{U}$ such that $\mathbf{S} = \mathbf{U}\Lambda_S\mathbf{U}^{\mathsf{H}}$ and $\mathbf{T} = \mathbf{U}\Lambda_T\mathbf{U}^{\mathsf{H}}$. Therefore,

$$\begin{aligned}
\mathbf{T}\mathbf{S} &= \mathbf{U}\Lambda_T\underbrace{\mathbf{U}^{\mathsf{H}}\mathbf{U}}_{\mathbf{I}}\Lambda_S\mathbf{U}^{\mathsf{H}} \\
&= \mathbf{U}\Lambda_T\Lambda_S\mathbf{U}^{\mathsf{H}}.
\end{aligned}$$

We conclude that $\mathbf{TS}$ has the same eigenvectors as $\mathbf{T}$ and $\mathbf{S}$ but with eigenvalues $\lambda_{TS} = \lambda_T \lambda_S$.

At this point it is instructive to go back to the counter example in the previous point and realize that the order of eigenvectors w.r.t. the order of eigenvalues matters.

4. (a) Spectral decomposition for circulant matrices: $\mathbf{C} = \mathbf{F}\mathbf{\Lambda}\mathbf{F}^{\mathsf{H}}$ where

$$\mathbf{F} = [\mathbf{f}_1, \ldots, \mathbf{f}_n], \mathbf{f}_k = \frac{1}{\sqrt{n}} \begin{bmatrix} \beta^{k0} \\ \beta^{k1} \\ \vdots \\ \beta^{k(n-1)} \end{bmatrix}, \beta = e^{\mathrm{i}2\pi/n}, \lambda_k = \sqrt{n}\mathbf{f}_k^{\mathsf{H}}\mathbf{c},$$

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}.$$

Proof:

$$\begin{aligned}
(\mathbf{C}\mathbf{f}_k)_l &= \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} c_{[l-m]} \beta^{km} \quad (\text{where } [l-m] = (l-m) \bmod n) \\
&= \frac{1}{\sqrt{n}} (c_0 \beta^{kl} + c_1 \beta^{k[l-1]} + \ldots + c_{n-1} \beta^{k[l-(n-1)]}) \\
&= \frac{1}{\sqrt{n}} \sum_{m'}^{n-1} c_{m'} \beta^{k[l-m']} \\
&= \frac{1}{\sqrt{n}} \sum_{m'}^{n-1} c_{m'} \beta^{k(l-m')} \\
&= \frac{\beta^{kl}}{\sqrt{n}} \sum_{m'=1}^{n-1} c_{m'} \beta^{-km'} \\
&= \lambda_k \frac{\beta^{kl}}{\sqrt{n}} = \lambda_k (\mathbf{f}_k)_l
\end{aligned}$$

Therefore, $\mathbf{C}\mathbf{f}_k = \lambda_k \mathbf{f}_k$, where $\lambda_k = \sqrt{n}\mathbf{f}_k^{\mathsf{H}}\mathbf{c}$. We see that all circulant matrices have the same eigenvectors.

Note: The fourth equality follows from

$$a \bmod n = a - \left\lfloor \frac{a}{n} \right\rfloor n$$

$$\Rightarrow \beta^{k[l-m']} = \beta^{k((l-m') - \lfloor \frac{l-m'}{n} \rfloor n)} = \beta^{k(l-m')} \underbrace{\beta^{-k \lfloor \frac{l-m'}{n} \rfloor n}}_{=1}.$$

(b) $\mathbf{C}_1 = \mathbf{F}\mathbf{\Lambda}_1\mathbf{F}^{\mathsf{H}}$ and $\mathbf{C}_2 = \mathbf{F}\mathbf{\Lambda}_2\mathbf{F}^{\mathsf{H}}$. Note that $\mathbf{F}$ is unitary since its columns are orthonormal.

$$\mathbf{C}_1\mathbf{C}_2 = \mathbf{F}\mathbf{\Lambda}_1\mathbf{\Lambda}_2\mathbf{F}^{\mathsf{H}}$$
$$\mathbf{C}_2\mathbf{C}_1 = \mathbf{F}\mathbf{\Lambda}_2\mathbf{\Lambda}_1\mathbf{F}^{\mathsf{H}}$$

Since diagonal matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ commute $\Rightarrow \mathbf{C}_1\mathbf{C}_2 = \mathbf{C}_2\mathbf{C}_1$.

Problem 2: Bandpass filter and orthogonal complement

Let

$$\mathcal{A} \triangleq \left\{\{a_k\}_{k\in\mathbb{Z}} \in l^2 : \widehat{a}(f) = 0 \text{ for all } f \in [-1/2, 1/2] \setminus [f_1, f_2]\right\}$$
$$\mathcal{B} \triangleq \left\{\{b_k\}_{k\in\mathbb{Z}} \in l^2 : \widehat{b}(f) = 0 \text{ for all } f \in [f_1, f_2]\right\},$$

where $\widehat{a}(f) = \sum_{k=-\infty}^{\infty} a_k e^{-\mathrm{i}2\pi kf}$ and $\widehat{b}(f) = \sum_{k=-\infty}^{\infty} b_k e^{-\mathrm{i}2\pi kf}$ denote the DTFTs of $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$, respectively. We need to show that $\mathcal{A}^\perp = \mathcal{B}$.

First we show that $\mathcal{A}^\perp \subset \mathcal{B}$. Indeed, take $\{c_k\}_{k\in\mathbb{Z}} \in \mathcal{A}^\perp$. By definition of the orthogonal complement, $\{c_k\}_{k\in\mathbb{Z}}$ is orthogonal to every sequence in $\mathcal{A}$, i.e.,

$$\langle\{c_k\}_{k\in\mathbb{Z}}, \{a_k\}_{k\in\mathbb{Z}}\rangle = 0 \text{ for all } \{a_k\}_{k\in\mathbb{Z}} \in \mathcal{A}. \tag{1}$$

Using Parseval's theorem, we compute the inner product $\langle\{c_k\}_{k\in\mathbb{Z}}, \{a_k\}_{k\in\mathbb{Z}}\rangle$ in the frequency domain as follows

$$\langle\{c_k\}_{k\in\mathbb{Z}}, \{a_k\}_{k\in\mathbb{Z}}\rangle = \sum_{k=-\infty}^{\infty} c_k a_k^* = \int_{-1/2}^{1/2} \widehat{c}(f)\, \widehat{a}^*(f) df,$$

where $\widehat{c}(f) = \sum_{k=-\infty}^{\infty} c_k e^{-\mathrm{i}2\pi kf}$. By definition of $\mathcal{A}$, it therefore follows that condition (1) can only be satisfied if $\widehat{c}(f) = 0$ for all $f \in [f_1, f_2]$. Therefore, by definition of $\mathcal{B}$, $\{c_k\}_{k\in\mathbb{Z}} \in \mathcal{B}$, which implies $\mathcal{A}^\perp \subset \mathcal{B}$.

Next, we show that $\mathcal{B} \subset \mathcal{A}^\perp$. Take $\{c_k\}_{k\in\mathbb{Z}} \in \mathcal{B}$. For every $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{A}$, using Parseval's theorem, we have

$$\langle\{c_k\}_{k\in\mathbb{Z}}, \{a_k\}_{k\in\mathbb{Z}}\rangle = \sum_{k=-\infty}^{\infty} c_k a_k^* = \int_{-1/2}^{1/2} \widehat{c}(f)\, \widehat{a}^*(f) df = 0,$$

where the last equality follows because $\widehat{a}(f)$ and $\widehat{c}(f)$ are supported on disjoint intervals. Therefore, $\{c_k\}_{k\in\mathbb{Z}} \in \mathcal{A}^\perp$, which implies $\mathcal{B} \subset \mathcal{A}^\perp$. By $\mathcal{A}^\perp \subset \mathcal{B}$ we therefore get $\mathcal{A}^\perp \subset \mathcal{B} \subset \mathcal{A}^\perp$ and hence $\mathcal{A}^\perp = \mathcal{B}$.

Problem 3: Tight frames

1. Take $\mathbf{x} = [x_0 \cdots x_{M-1}]^\mathsf{T} \in \mathbb{C}^M$ and compute

$$\begin{aligned}
\sum_{k=0}^{KM-1} |\langle\mathbf{x}, \mathbf{g}_k\rangle|^2 &= \sum_{k=0}^{KM-1} \left|\sum_{n=0}^{M-1} x_n e^{-\mathrm{i}2\pi kn/(KM)}\right|^2 \\
&= \sum_{k=0}^{KM-1} \sum_{n=0}^{M-1} \sum_{n'=0}^{M-1} x_n x_{n'}^* e^{-\mathrm{i}2\pi kn/(KM)} e^{\mathrm{i}2\pi kn'/(KM)} \\
&= \sum_{n=0}^{M-1} \sum_{n'=0}^{M-1} x_n x_{n'}^* \underbrace{\sum_{k=0}^{KM-1} e^{\frac{\mathrm{i}2\pi k}{KM}(n'-n)}}_{=\begin{cases} KM, & n = n' \\ 0, & n \neq n' \end{cases}} \\
&= \sum_{n=0}^{M-1} KM|x_n|^2 = KM\|\mathbf{x}\|^2.
\end{aligned} \tag{1}$$

Since (1) holds for all $\mathbf{x} \in \mathbb{C}^M$, $\{\mathbf{g}_k\}_{0 \le k \le KM-1}$ is a frame for $\mathbb{C}^M$ with upper and lower frame bound equal to KM, i.e., it is a tight frame with frame bound $A = KM$.

2. Take $x(t) \in \mathcal{L}^2([0,T])$ and compute

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} |\langle x, g_k\rangle|^2 &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{-\mathrm{i}2\pi kt/T} dt \int_{-\infty}^{\infty} x^*(t') e^{\mathrm{i}2\pi kt'/T} dt' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, x^*(t') \underbrace{\sum_{k=-\infty}^{\infty} e^{\mathrm{i}2\pi k \frac{(t'-t)}{T}}}_{=T\sum_{k=-\infty}^{\infty}\delta(t'-t-kT)} dt' dt \\
&\overset{(a)}{=} \sum_{k=-\infty}^{\infty} T \int_{-\infty}^{\infty} x(t)\, x^*(t+kT) dt \\
&\overset{(b)}{=} \sum_{k=-\infty}^{\infty} T \int_{0}^{T} x(t)\, x^*(t+kT) dt \\
&\overset{(c)}{=} T \int_{0}^{T} x(t)\, x^*(t) dt \\
&\overset{(d)}{=} T\,\langle x, x\rangle = T\|x\|^2, \qquad (2)
\end{aligned}
$$

where in (a) we used the sifting property of the δ-function and in (b), (c), and (d) we used that $x(t) \in \mathcal{L}^2([0,T])$. Since (2) holds for all $x(t) \in \mathcal{L}^2([0,T])$, $\{g_k(t)\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{L}^2([0,T])$ with upper and lower frame bound equal to T, i.e., it is a tight frame with frame bound $A = T$.

Problem 4: Sampling theory I

1. Take $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$. By definition of $\mathbb{A}$, we have

$$\mathbb{A}\{a_k\}_{k\in\mathbb{Z}} = \sum_{k=-\infty}^{\infty} a_k h_{\mathrm{LP}}\left(\frac{t}{T} - k\right).$$

The result now follows immediately by noting that

$$\tilde{\mathbb{T}}^{\dagger}\{a_k\}_{k\in\mathbb{Z}} = \sum_{k=-\infty}^{\infty} a_k \tilde{g}_k(t)$$

where $\tilde{g}_k(t) = T g_k(t) = h_{\mathrm{LP}}\left(\frac{t}{T} - k\right)$.

2. Take $\{b_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})^\perp$. Then,

$$\begin{aligned}
\mathbb{A}\{b_k\}_{k\in\mathbb{Z}} &= \sum_{k=-\infty}^{\infty} b_k h_{\mathrm{LP}}\left(\frac{t}{T}-k\right) \\
&= \sum_{k=-\infty}^{\infty} b_k \int_{-1/2}^{1/2} \widehat{h}_{\mathrm{LP}}(f) e^{\mathrm{i}2\pi f(t/T-k)} df \\
&= \int_{-1/2}^{1/2} \widehat{h}_{\mathrm{LP}}(f) e^{\mathrm{i}2\pi t f/T} \underbrace{\sum_{k=-\infty}^{\infty} b_k e^{-\mathrm{i}2\pi k f}}_{\widehat{b}(f)} df \\
&= \int_{-1/2}^{1/2} \widehat{b}(f)\widehat{h}_{\mathrm{LP}}(f) e^{\mathrm{i}2\pi t f/T} df \\
&= 0, \qquad (1)
\end{aligned}$$

where the last equality follows because $\widehat{b}(f)$ is supported on the set $[-1/2,-BT]\cup[BT,1/2]$ and $\widehat{h}_{\mathrm{LP}}(f)$ is supported on the set $[-BT,BT]$.

3. Take $\{c_k\}_{k\in\mathbb{Z}} \in l^2$. We can write $\{c_k\}_{k\in\mathbb{Z}} = \{a_k\}_{k\in\mathbb{Z}} + \{b_k\}_{k\in\mathbb{Z}}$ with $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$ and $\{b_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})^\perp$. It was proven in **1.** and **2.** above that $\mathbb{A}\{a_k\}_{k\in\mathbb{Z}} = \tilde{\mathbb{T}}^\dagger\{a_k\}_{k\in\mathbb{Z}}$ and $\mathbb{A}\{b_k\}_{k\in\mathbb{Z}} = 0$. Therefore,

$$\mathbb{A}\{c_k\}_{k\in\mathbb{Z}} = \mathbb{A}\{a_k\}_{k\in\mathbb{Z}} + \underbrace{\mathbb{A}\{b_k\}_{k\in\mathbb{Z}}}_{0} = \tilde{\mathbb{T}}^\dagger\{a_k\}_{k\in\mathbb{Z}}.$$

Since $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$ and $\mathbb{P}$ is the orthogonal projection operator onto $\mathcal{R}(\mathbb{T})$, we have $\mathbb{P}\{a_k\}_{k\in\mathbb{Z}} = \{a_k\}_{k\in\mathbb{Z}}$. Similarly, since $\{b_k\}_{k\in\mathbb{Z}}$ is in the orthogonal complement of $\mathcal{R}(\mathbb{T})$, we have $\mathbb{P}\{b_k\}_{k\in\mathbb{Z}} = 0$. Therefore,

$$\tilde{\mathbb{T}}^\dagger\mathbb{P}\{c_k\}_{k\in\mathbb{Z}} = \tilde{\mathbb{T}}^\dagger \underbrace{\mathbb{P}\{a_k\}_{k\in\mathbb{Z}}}_{\{a_k\}_{k\in\mathbb{Z}}} + \tilde{\mathbb{T}}^\dagger \underbrace{\mathbb{P}\{b_k\}_{k\in\mathbb{Z}}}_{0} = \tilde{\mathbb{T}}^\dagger\{a_k\}_{k\in\mathbb{Z}}.$$

We conclude that $\mathbb{A} = \tilde{\mathbb{T}}^\dagger\mathbb{P}$, as required.

Problem 5: Sampling theory II

The proof is accomplished in three steps.

1. Take $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})^\perp$. We have,

$$\begin{aligned}
\mathbb{B}\{a_k\}_{k\in\mathbb{Z}} &\overset{(a)}{=} \sum_{k=-\infty}^{\infty} a_k h_{\mathrm{out}}\left(\frac{t}{T}-k\right) \\
&\overset{(b)}{=} \int_{-1/2}^{1/2} \widehat{a}(f)\widehat{h}_{\mathrm{out}}(f) e^{\mathrm{i}2\pi t f/T} df \\
&\overset{(c)}{=} \int_{-1/2}^{-BT} \widehat{a}(f)\mathrm{arb}(f) e^{\mathrm{i}2\pi t f/T} df + \int_{BT}^{1/2} \widehat{a}(f)\mathrm{arb}(f) e^{\mathrm{i}2\pi t f/T} df,
\end{aligned}$$

where (a) follows by definition of $\mathbb{B}$, (b) follows by the same steps as the solution of item 2 in Problem 4, and in (c) we used that $\widehat{a}(f) = 0$ for $f \in [-BT, BT]$. Similarly,

$$\begin{aligned}\mathbb{M}\{a_k\}_{k\in\mathbb{Z}} &\overset{(a)}{=} \sum_{k=-\infty}^{\infty} a_k h_M\left(\frac{t}{T} - k\right) \\ &\overset{(b)}{=} \int_{-1/2}^{1/2} \widehat{a}(f)\widehat{h}_M(f)e^{\mathrm{i}2\pi tf/T}df \\ &\overset{(c)}{=} \int_{-1/2}^{-BT} \widehat{a}(f)\mathrm{arb}(f)e^{\mathrm{i}2\pi tf/T}df + \int_{BT}^{1/2} \widehat{a}(f)\mathrm{arb}(f)e^{\mathrm{i}2\pi tf/T}df,\end{aligned}$$

where (a) follows by definition of $\mathbb{M}$, (b) follows by the same steps as in in solution of item 2 in Problem 4, and in (c) we used that $\widehat{a}(f) = 0$ for $f \in [-BT, BT]$. We conclude that $\mathbb{B}\{a_k\}_{k\in\mathbb{Z}} = \mathbb{M}\{a_k\}_{k\in\mathbb{Z}}$.

2. Next, take $\{b_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$. Then,

$$\begin{aligned}\mathbb{B}\{b_k\}_{k\in\mathbb{Z}} &= \sum_{k=-\infty}^{\infty} b_k h_{\mathrm{out}}\left(\frac{t}{T} - k\right) \\ &= \int_{-1/2}^{1/2} \widehat{b}(f)\widehat{h}_{\mathrm{out}}(f)e^{\mathrm{i}2\pi tf/T}df \\ &= 0,\end{aligned}$$

where the third equality follows because $\widehat{b}(f)$ is supported on the set $[-BT, BT]$ and $\widehat{h}_{\mathrm{out}}(f)$ is supported on the set $[-1/2, -BT] \cup [BT, 1/2]$.

3. Finally, take $\{c_k\}_{k\in\mathbb{Z}} \in l^2$. We can write $\{c_k\}_{k\in\mathbb{Z}} = \{a_k\}_{k\in\mathbb{Z}} + \{b_k\}_{k\in\mathbb{Z}}$ with $\{a_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})^\perp$ and $\{b_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$. It was shown in item 1 above that $\mathbb{B}\{a_k\}_{k\in\mathbb{Z}} = \mathbb{M}\{a_k\}_{k\in\mathbb{Z}}$ and in item 2 above that $\mathbb{B}\{b_k\}_{k\in\mathbb{Z}} = 0$ so that

$$\mathbb{B}\{c_k\}_{k\in\mathbb{Z}} = \mathbb{B}\{a_k\}_{k\in\mathbb{Z}} + \underbrace{\mathbb{B}\{b_k\}_{k\in\mathbb{Z}}}_{0} = \mathbb{M}\{a_k\}_{k\in\mathbb{Z}}. \tag{1}$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is in the orthogonal complement of $\mathcal{R}(\mathbb{T})$ and $\mathbb{P}$ is the orthogonal projection operator onto $\mathcal{R}(\mathbb{T})$, we have $\mathbb{P}\{a_k\}_{k\in\mathbb{Z}} = 0$, or, equivalently, $(\mathbb{I}_{l^2} - \mathbb{P})\{a_k\}_{k\in\mathbb{Z}} = \{a_k\}_{k\in\mathbb{Z}}$. Similarly, since $\{b_k\}_{k\in\mathbb{Z}} \in \mathcal{R}(\mathbb{T})$, we have $\mathbb{P}\{b_k\}_{k\in\mathbb{Z}} = \{b_k\}_{k\in\mathbb{Z}}$, or, equivalently $(\mathbb{I}_{l^2} - \mathbb{P})\{b_k\}_{k\in\mathbb{Z}} = 0$. Therefore,

$$\mathbb{M}(\mathbb{I}_{l^2} - \mathbb{P})\{c_k\}_{k\in\mathbb{Z}} = \mathbb{M}\,\underbrace{(\mathbb{I}_{l^2} - \mathbb{P})\{a_k\}_{k\in\mathbb{Z}}}_{\{a_k\}_{k\in\mathbb{Z}}} + \mathbb{M}\,\underbrace{(\mathbb{I}_{l^2} - \mathbb{P})\{b_k\}_{k\in\mathbb{Z}}}_{0} = \mathbb{M}\{a_k\}_{k\in\mathbb{Z}}. \tag{2}$$

Comparing (1) and (2), we can therefore conclude that $\mathbb{B} = \mathbb{M}(\mathbb{I}_{l^2} - \mathbb{P})$, as required.

Problem 6: Weyl-Heisenberg (WH) frame in finite dimensions

The solution for the numerical part of the problem can be found in `wh_frames.ipynb` file.

1. The analysis operator $\mathbb{T} : \mathbb{C}^M \to \mathbb{C}^{KL}$ maps a vector $\mathbf{x} = [x[0] \cdots x[M-1]]^\mathsf{T} \in \mathbb{C}^M$ to the sequence of inner products

$$\mathbb{T}\mathbf{x} = \{\langle \mathbf{x}, \mathbf{g}_{k,l} \rangle\}_{k=0,\ldots,K-1, l=0,\ldots,L-1}.$$

It is convenient to arrange the inner products $\{\langle \mathbf{x}, \mathbf{g}_{k,l} \rangle\}_{k=0,\ldots,K-1, l=0,\ldots,L-1}$ into a vector $\mathbf{y} = [y[0] \cdots y[KL-1]]^\mathsf{T} \in \mathbb{C}^{KL}$ as follows:

$$y[Lk+l] = \langle \mathbf{x}, \mathbf{g}_{k,l} \rangle = \sum_{n=0}^{M-1} x[n] g_{k,l}^*[n], \; k = 0, \ldots, K-1, \; l = 0, \ldots, L-1. \tag{1}$$

Next define the $KL \times M$ matrix $\mathbf{T}$ according to

$$\mathbf{T}_{Lk+l,n} \triangleq g_{k,l}^*[n] = g^*[(n-lT) \mod M] e^{-2\pi i kn/K},$$

where $\mathbf{T}_{i,j}$ denotes the element in the ith row and jth column of $\mathbf{T}$. Now we can rewrite (1) as $\mathbf{y} = \mathbf{T}\mathbf{x}$, which means that the analysis operator $\mathbb{T}$ is represented by the matrix $\mathbf{T}$.

2. Since the analysis operator $\mathbb{T}$ is represented by the matrix $\mathbf{T}$, the adjoint operator $\mathbb{T}^\dagger$ is represented by the Hermitian transpose of the matrix $\mathbf{T}$, i.e., by $\mathbf{T}^\mathsf{H}$.

3. Since the frame operator is given by $\mathbb{S} = \mathbb{T}^\dagger\mathbb{T}$, it is represented by the matrix $\mathbf{S} = \mathbf{T}^\mathsf{H}\mathbf{T}$. The element in the nth row and mth column of $\mathbf{S}$ can be found as follows:

$$\begin{aligned}
\mathbf{S}_{n,m} &= \sum_{k=0}^{K-1}\sum_{l=0}^{L-1} (\mathbf{T}_{Lk+l,n})^* \mathbf{T}_{Lk+l,m} \\
&= \sum_{k=0}^{K-1}\sum_{l=0}^{L-1} g[(n-lT) \mod M] e^{2\pi i kn/K} \, g^*[(m-lT) \mod M] e^{-2\pi i km/K} \\
&= \sum_{l=0}^{L-1} g[(n-lT) \mod M] \, g^*[(m-lT) \mod M] \sum_{k=0}^{K-1} e^{2\pi i k(n-m)/K} \\
&= \begin{cases} K \sum_{l=0}^{L-1} g[(n-lT) \mod M] \, g^*[(m-lT) \mod M], & \text{if } (n-m)/K \in \mathbb{Z} \\ 0, & \text{if } (n-m)/K \notin \mathbb{Z}. \end{cases}
\end{aligned} \tag{2}$$

4. The lower frame bound and the upper frame bound are given by the smallest and the largest eigenvalue of $\mathbf{S}$, respectively. To check that the lower frame bound is strictly positive, we therefore need to verify numerically that all eigenvalues of $\mathbf{S}$ are strictly positive. Since the number of elements in the set $\{\mathbf{g}_{k,l}\}_{k=0,\ldots,K-1, l=0,\ldots,L-1}$ is finite and each vector has bounded entries, the upper frame bound is finite.

5. For $K = M$ and $n, m \in \{0, \ldots, M-1\}$ we have that

$$\frac{n-m}{K} \in \mathbb{Z} \Leftrightarrow n = m.$$

Therefore, $\mathbf{S}$ is a diagonal matrix with main diagonal entries

$$\mathbf{S}_{n,n} = M \sum_{l=0}^{M-1} g[(n-l) \mod M] \, g^*[(n-l) \mod M] = M\|g\|^2.$$

The eigenvalues of $\mathbf{S}$ are therefore all equal to $M\|g\|^2$. Since $\|g\| \neq 0$, we see that $\{\mathbf{g}_{k,l}\}_{k=0,\ldots,K-1, l=0,\ldots,L-1}$ is a tight frame for $\mathbb{C}^M$ with $A = B = M\|g\|^2$.

6. Vary the parameters (T, K) and, for each parameter pair, compute the eigenvalues of $\mathbf{S}$. The frame bounds are given by the smallest and the largest eigenvalue of $\mathbf{S}$. When all eigenvalues of $\mathbf{S}$ are strictly positive you have a frame. If at least one eigenvalue of $\mathbf{S}$ is equal to zero, $\{\mathbf{g}_{k,l}\}_{k=0,\ldots,K-1,l=0,\ldots,L-1}$ is not a frame for $\mathbb{C}^M$.

7. We have shown that $\{\mathbf{g}_{k,l}\}_{k=0,\ldots,K-1,l=0,\ldots,L-1}$ is a tight frame with frame bound $M\|g\|^2$. This implies that the frame operator is given by the matrix $\mathbf{S} = M\|\mathbf{g}\|^2\mathbf{I}_M$. The canonical dual frame of $\{\mathbf{g}_{k,l}\}_{k=0,\ldots,K-1,l=0,\ldots,L-1}$ is then given by $\{\tilde{\mathbf{g}}_{k,l}\}_{k=0,\ldots,K-1,l=0,\ldots,L-1}$ with

$$\tilde{\mathbf{g}}_{k,l} = \mathbf{S}^{-1}\mathbf{g}_{k,l} = \frac{\mathbf{g}_{k,l}}{M\|\mathbf{g}\|^2}.$$

   As a consequence, $\tilde{\mathbf{g}}_{k,l}$ can be written as:

$$\tilde{g}_{k,l}[n] = \tilde{g}[(n-l) \mod M]e^{2\mathrm{i}\pi kn/K}, \; k = 0,\ldots,M-1, l = 0,\ldots,L-1, n = 0,\ldots,M-1$$

   where we defined the dual prototype $\tilde{\mathbf{g}} = \mathbf{S}^{-1}\mathbf{g} = \frac{\mathbf{g}}{M\|\mathbf{g}\|^2}$.

Problem 7: Frame expansion with noise

We have the following:

$$\begin{aligned}
\mathbb{E}\{\|\mathbf{f} - \mathbf{f}_{\mathrm{w}}\|^2\} &= \mathbb{E}\left\{\|\frac{1}{A}\sum_{j=1}^{M}(\langle\mathbf{f},\mathbf{g}_j\rangle\,\mathbf{g}_j - \langle\mathbf{f},\mathbf{g}_j\rangle\,\mathbf{g}_j - w_j\mathbf{g}_j)\|^2\right\} \\
&= \mathbb{E}\left\{\|\frac{1}{A}\sum_{j=1}^{M} w_j\mathbf{g}_j\|^2\right\} \\
&= \mathbb{E}\left\{\frac{1}{A^2}\sum_{j=1}^{M}\sum_{k=1}^{M} w_j w_k^*\,\langle\mathbf{g}_j,\mathbf{g}_k\rangle\right\} \\
&= \frac{1}{A^2}\sum_{j=1}^{M}\sum_{k=1}^{M}\underbrace{\mathbb{E}\{w_j w_k^*\}}_{\sigma^2\delta_{jk}}\langle\mathbf{g}_j,\mathbf{g}_k\rangle \\
&= \frac{\sigma^2}{A^2}\sum_{j=1}^{M}\|\mathbf{g}_j\|^2 \\
&= \frac{\sigma^2 NA}{A^2} = \frac{\sigma^2 N}{r}.
\end{aligned}$$

The MSE is inversely proportional to the redundancy. Therefore, it is an advantage to formulate algorithms involving frames than bases, which have redundancy 1.

Problem 8: Weyl-Heisenberg frame

As suggested in the problem statement, we start by observing that $\widehat{\mathbb{T}_{l/2}\mathbb{M}_k\phi} = \mathbb{M}_{-l/2}\mathbb{T}_k\hat{\phi}$. Indeed, we have the following for all $\nu \in \mathbb{R}$:

$$\begin{aligned}
\widehat{\mathbb{T}_{l/2}\mathbb{M}_k}\phi &= \int_{-\infty}^{\infty} \mathbb{T}_{l/2}\mathbb{M}_k\phi(t)e^{-2\pi i\nu t}dt \\
&= \int_{-\infty}^{\infty} e^{2\pi i k(t-l/2)}\phi(t-l/2)e^{-2\pi i\nu t}dt \\
&= \int_{-\infty}^{\infty} e^{2\pi i k t'}\phi(t')e^{-2\pi i\nu(t'+l/2)}dt' \\
&= e^{-2\pi i\nu l/2}\int_{-\infty}^{\infty} \phi(t')e^{-2\pi i(\nu-k)t'}dt' \\
&= e^{-2\pi i\nu l/2}\hat{\phi}(\nu-k) \\
&= \mathbb{M}_{-l/2}\mathbb{T}_k\hat{\phi}.
\end{aligned}$$

Using Parseval's equality, we can then write

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle f, \mathbb{T}_{l/2}\mathbb{M}_k\Psi\right\rangle\right|^2 &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle f, \frac{1}{\sqrt{2}}\mathbb{T}_{l/2}\mathbb{M}_k\phi\right\rangle\right|^2 \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle \hat{f}, \frac{1}{\sqrt{2}}\widehat{\mathbb{T}_{l/2}\mathbb{M}_k}\phi\right\rangle\right|^2 \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle \hat{f}, \frac{1}{\sqrt{2}}\mathbb{M}_{-l/2}\mathbb{T}_k\hat{\phi}\right\rangle\right|^2 \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\int_{-\infty}^{\infty} \hat{f}(\nu)\frac{1}{\sqrt{2}}e^{-2\pi i\frac{l}{2}\nu}\mathbb{T}_k\hat{\phi}(\nu)d\nu\right|^2.
\end{aligned}$$

Since supp $\hat{\phi} \subset [-1, 1]$, we have that supp $\mathbb{T}_k\hat{\phi} \subset [k-1, k+1]$, which gives

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle f, \mathbb{T}_{l/2}\mathbb{M}_k\Psi\right\rangle\right|^2 = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\int_{k-1}^{k+1} \hat{f}(\nu)\frac{1}{\sqrt{2}}e_{l/2}(\nu)^*\mathbb{T}_k\hat{\phi}(\nu)d\nu\right|^2,$$

where $e_{l/2}(\nu) = e^{2\pi i\frac{l}{2}\nu}$ for all $\nu \in [k-1, k+1]$. We recognize the standard inner product of $L^2[k-1, k+1]$:

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle f, \mathbb{T}_{l/2}\mathbb{M}_k\Psi\right\rangle\right|^2 = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle \hat{f}\mathbb{T}_k\hat{\phi}, \frac{1}{\sqrt{2}}e_{l/2}\right\rangle_{L^2[k-1,k+1]}\right|^2.$$

Since $\{e_{l/2}/\sqrt{2}\}_{l\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[k-1, k+1]$ for all $k \in \mathbb{Z}$, we can use again Parseval's equality to write that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle f, \mathbb{T}_{l/2}\mathbb{M}_k\Psi\right\rangle\right|^2 &= \sum_{k\in\mathbb{Z}} \|\hat{f}\mathbb{T}_k\hat{\phi}\|^2_{L^2[k-1,k+1]} \\
&= \sum_{k\in\mathbb{Z}} \int_{k-1}^{k+1} \left|\hat{f}(\nu)\right|^2\left|\hat{\phi}(\nu-k)\right|^2 d\nu.
\end{aligned}$$

Since $\operatorname{supp}\mathbb{T}_k\hat{\phi} \subset [k-1, k+1]$ and $\sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\nu - k)\right|^2 = 1$ for all $\nu \in \mathbb{R}$, it holds that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \left|\left\langle f, \mathbb{T}_{l/2}\mathbb{M}_k\Psi\right\rangle\right|^2 &= \sum_{k\in\mathbb{Z}} \int_{k-1}^{k+1} \left|\hat{f}(\nu)\right|^2 \left|\hat{\phi}(\nu-k)\right|^2 d\nu \\
&= \int_{k-1}^{k+1} \left|\hat{f}(\nu)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\nu-k)\right|^2 d\nu \qquad (1)\\
&= \int_{k-1}^{k+1} \left|\hat{f}(\nu)\right|^2 d\nu \\
&= \|\hat{f}\|^2 = \|f\|^2. \qquad (2)
\end{aligned}$$

Note that we can exchange the order of the summation and the integral in (1), since the series $\int_{k-1}^{k+1} |\hat{f}(\nu)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\nu-k)|^2 d\nu$ converges. The equality in (2) shows that $\{\mathbb{T}_{l/2}\mathbb{M}_k\Psi\}$ forms a tight frame with frame bound $A = 1$.

Problem 9: Wavelet frame

Let $f \in L^2(\mathbb{R})$. Using Parseval's theorem, we have that

$$\begin{aligned}
\sum_{j\geq -1}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 &= \sum_{k\in\mathbb{Z}} |\langle f, \psi_{-1,k}\rangle|^2 + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|\left\langle \hat{f}, \widehat{\psi_{-1,k}}\right\rangle\right|^2 + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \left|\left\langle \hat{f}, \widehat{\psi_{j,k}}\right\rangle\right|^2.
\end{aligned}$$

By definition of $\Psi_{-1,k}$ and $\Psi_{j,k}$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$, it holds that for all $\nu \in \mathbb{R}$

$$\begin{aligned}
&\widehat{\psi_{-1,k}}(\nu) = \hat{\phi}(\nu)e^{-2\pi i\nu k/2}/\sqrt{2} = \hat{\phi}(\nu)e_{-k/2}(\nu)/\sqrt{2} \\
&\widehat{\psi_{j,k}}(\nu) = 2^{-j/2-1}\hat{\phi}(\nu)e^{-2\pi i\nu k 2^{-j}/4} = 2^{-j/2-1}\hat{\psi}(2^{-j}\nu)e_{-k2^{-j}/4}(\nu),
\end{aligned}$$

where we defined $e_k(\nu) = e^{2\pi i k\nu}$ for all $\nu \in \mathbb{R}$. This gives

$$\begin{aligned}
\sum_{j\geq -1}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 &= \sum_{k\in\mathbb{Z}} \left|\left\langle \hat{f}, \frac{1}{\sqrt{2}}e_{-k/2}\hat{\phi}\right\rangle\right|^2 + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \left|\left\langle \hat{f}, 2^{-j/2-1}e_{-k2^{-j}/4}\hat{\psi}(2^{-j}\cdot)\right\rangle\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|\int_{-\infty}^{\infty} \hat{f}(\nu)\hat{\phi}(\nu)\frac{1}{\sqrt{2}}e_{k/2}(\nu)d\nu\right|^2 + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \left|\int_{-\infty}^{\infty} \hat{f}(\nu)\hat{\psi}(2^{-j}\nu)2^{-j/2-1}e_{k2^{-j}/4}(\nu)d\nu\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|\int_{-1}^{1} \hat{f}(\nu)\hat{\phi}(\nu)\frac{1}{\sqrt{2}}e_{k/2}(\nu)d\nu\right|^2 + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \left|\int_{Q_j} \hat{f}(\nu)\hat{\psi}(2^{-j}\nu)2^{-j/2-1}e_{k2^{-j}/4}(\nu)d\nu\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|\left\langle \hat{f}\hat{\phi}, \frac{1}{\sqrt{2}}e_{k/2}\right\rangle\right|^2 + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \left|\left\langle \hat{f}\hat{\psi}(2^{-j}\cdot), 2^{-j/2-1}e_{k2^{-j}/4}\right\rangle_{L^2(Q_j)}\right|^2
\end{aligned}$$

since $\operatorname{supp}\hat{\phi} \subset [-1, 1]$ and $\hat{\psi}(2^{-j}\cdot) \subset Q_j \triangleq [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}]$. We have seen in the previous problem that $\{e_{k/2}/\sqrt{2}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[-1, 1]$.

Likewise we have that $\{2^{-j/2-1}e_{k2^{-j}/4}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for Q_j. Thus, we can use

Parseval's equality to write that

$$
\begin{aligned}
\sum_{j \geq -1} \sum_{k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 &= \|\hat{f}\hat{\phi}\|^2_{L^2[-1,1]} + \sum_{j \in \mathbb{N}} \|\hat{f}\hat{\psi}(2^{-j}\cdot)\|^2_{L^2(Q_j)} \\
&= \int_{-1}^{1} \left|\hat{f}(\nu)\right|^2 \left|\hat{\phi}(\nu)\right|^2 d\nu + \sum_{j \in \mathbb{N}} \int_{Q_j} \left|\hat{f}(\nu)\right|^2 \left|\hat{\psi}(2^{-j}\nu)\right|^2 d\nu \\
&= \int_{-\infty}^{\infty} \left|\hat{f}(\nu)\right|^2 \left|\hat{\phi}(\nu)\right|^2 d\nu + \sum_{j \in \mathbb{N}} \int_{-\infty}^{\infty} \left|\hat{f}(\nu)\right|^2 \left|\hat{\psi}(2^{-j}\nu)\right|^2 d\nu,
\end{aligned}
$$

using again the fact that $\mathcal{T}\,\hat{\phi} \subset [-1,1]$ and $\mathcal{T}\,\hat{\psi}(2^{-j}\cdot) \subset Q_j$. We can exchange the order between the summation and the integral and use the fact that $\left|\hat{\phi}(\nu)\right|^2 + \sum_{j \in \mathbb{N}} \left|\hat{\phi}(2^{-j}\nu)\right|^2 = 1$ for all $\nu \in \mathbb{R}$. This yields

$$
\begin{aligned}
\sum_{j \geq -1} \sum_{k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 &= \int_{-\infty}^{\infty} \left|\hat{f}(\nu)\right|^2 \left|\hat{\phi}(\nu)\right|^2 d\nu + \sum_{j \in \mathbb{N}} \int_{-\infty}^{\infty} \left|\hat{f}(\nu)\right|^2 \left|\hat{\psi}(2^{-j}\nu)\right|^2 d\nu \\
&= \int_{-\infty}^{\infty} \left|\hat{f}(\nu)\right|^2 \left(\left|\hat{\phi}(\nu)\right|^2 + \left|\hat{\psi}(2^{-j}\nu)\right|^2 \right) d\nu \\
&= \int_{-\infty}^{\infty} \left|\hat{f}(\nu)\right|^2 d\nu = \|\hat{f}\|^2 = \|f\|^2,
\end{aligned}
$$

which shows that $\{\psi_{j,k}\}_{j \geq -1, k \in \mathbb{Z}}$ forms a tight frame for $L^2(\mathbb{R})$ with frame bound 1.