Mathematical methods for machine learning and signal processing SS 19

Solutions to problem set 5

Prof. V. Morgenshtern

Solver: V. Morgenshtern

Problem 1: Nuclear norm ball

The singular values of $\mathbf{A}$ are the square roots of eigenvalues of

$$\mathbf{A}^\top\mathbf{A} = \begin{bmatrix} x^2+y^2 & xy+yz \\ xy+yz & z^2+y^2 \end{bmatrix}.$$

To find the eigenvalues of $\mathbf{A}^\top\mathbf{A}$, solve $\det(\mathbf{A}^\top\mathbf{A} - \lambda\mathbf{I}) = 0$.

$$\begin{aligned}
&(x^2+y^2-\lambda)(z^2+y^2-\lambda) - (xy+yz)^2 = 0 \\
&(x^2+y^2)(z^2+y^2) - \lambda(x^2+2y^2+z^2) + \lambda^2 - (xy+yz)^2 = 0
\end{aligned}$$

Solving and simplifying:

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{2}\left(x^2+2y^2+z^2 \pm \sqrt{(x^2+2y^2+z^2)^2 - 4\left((x^2+y^2)(y^2+z^2) - (xy+yz)^2\right)}\right) \\
&= \frac{1}{2}\left(x^2+2y^2+z^2 \pm \sqrt{(x-z)^2(x+z)^2 + 4y^2(x+z)^2}\right) \\
&= \frac{1}{2}\left(x^2+2y^2+z^2 \pm |x+z|\sqrt{(x-z)^2+4y^2}\right).
\end{aligned}$$

Now, $\|\mathbf{A}\|_* = 1$ is equivalent to $\sqrt{\lambda_1} + \sqrt{\lambda_2} = 1$ or $\lambda_1 + \lambda_2 + 2\sqrt{\lambda_1}\sqrt{\lambda_2} = 1$, which can be written as

$$x^2+2y^2+z^2 + \sqrt{(x^2+2y^2+z^2)^2 - (x+z)^2((x-z)^2+4y^2)} = 1.$$

Simplifying:

$$\begin{aligned}
x^2+2y^2+z^2 + \sqrt{4(y^2-xz)^2} &= 1 \\
x^2+2y^2+z^2 + 2\left|y^2-xz\right| &= 1.
\end{aligned}$$

If $y^2 > xz$, the equation simplifies to

$$(x-z)^2 + 4y^2 = 1. \tag{1}$$

Now we are ready to plot the solutions of this equation. See `nuclear.ipynb`

Problem 2: Basic matrix completion experiment

See `matrix_completion.ipynb`

Problem 3: Approximate invariance of scattering transform to translations

See `scattering_invariance.ipynb`

Problem 4: Learning in scattering domain via PCA

See `scattering_pca.ipynb`