

Solving Underdetermined Linear Systems with Highly Correlated Columns

Veniamin I. Morgenshtern

Statistics Department, Stanford

Problem: Estimate a **high-dimensional** object from (relatively) **few** corrupted data points.

“Big data” world

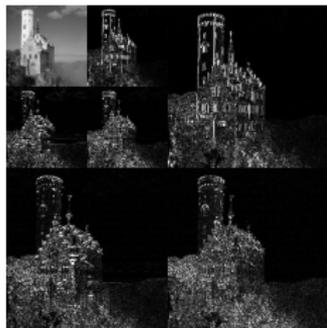
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Assumption: The object is **simple**: low-dimensional structure.

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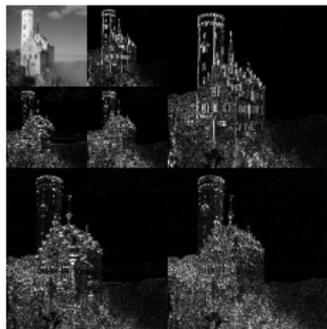


signal processing

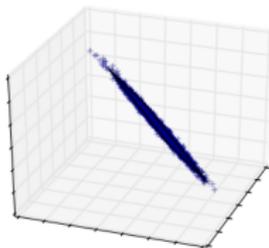
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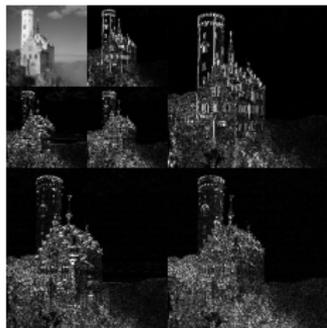


statistics

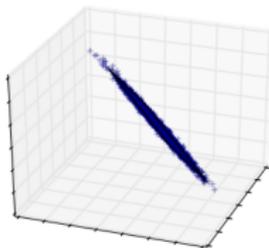
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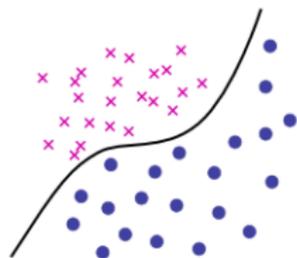
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signal processing



statistics



machine learning

Underdetermined linear system

$$\begin{array}{|c} y \end{array} = \begin{array}{|c|c|c|} \hline & A & \\ \hline \end{array} \begin{array}{|c} x \end{array}$$

Few linear measurements about a high-dimensional object

Underdetermined linear system

$$y = Ax$$

The diagram illustrates the equation $y = Ax$ using boxes to represent dimensions. On the left, a vertical box contains the variable y . This is followed by an equals sign. To the right of the equals sign is a horizontal box containing the letter A . This is followed by a vertical box containing the variable x . The boxes are drawn to represent the relative dimensions of the variables: y and x are tall and narrow, while A is wide and short.

Few linear measurements about a high-dimensional object

How can we possibly recover the object?

Underdetermined linear system

$$y = Ax$$

The diagram illustrates the equation $y = Ax$ using boxes to represent dimensions. The box for y is tall and narrow, representing a low-dimensional vector. The box for A is wide and short, representing a matrix with many columns and few rows. The box for x is tall and narrow, representing a high-dimensional vector. This visualizes the concept of an underdetermined system where the number of unknowns (x) is greater than the number of equations (y).

Few linear measurements about a high-dimensional object

How can we possibly recover the object?

The object has low-dimensional representation (sparsity)

Underdetermined linear system

$$y = Ax$$

The diagram illustrates the equation $y = Ax$ using boxes to represent dimensions. On the left, a vertical box contains the variable y . To its right is an equals sign. Further right is a horizontal box containing the matrix A . To the right of A is a vertical box containing the variable x . This visualizes that y is a low-dimensional vector, A is a matrix with more columns than rows, and x is a high-dimensional vector.

Few linear measurements about a high-dimensional object

How can we possibly recover the object?

The object has low-dimensional representation (sparsity)

Many problems are naturally of this form.

Underdetermined linear system

$$y = Ax$$

Few linear measurements about a high-dimensional object

How can we possibly recover the object?

The object has low-dimensional representation (sparsity)

Many problems are naturally of this form.

Even more problems can be forced into this form!

Huge range of “big data” applications



compressed sensing
MRI



model selection via
LASSO

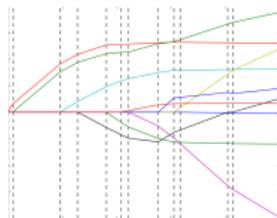


low rank matrix
recovery
(Netflix Prize)

Huge range of “big data” applications



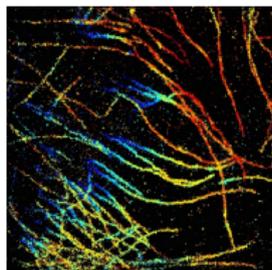
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super-resolution microscopy

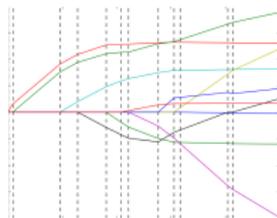


radar imaging

Huge range of “big data” applications



compressed sensing
MRI

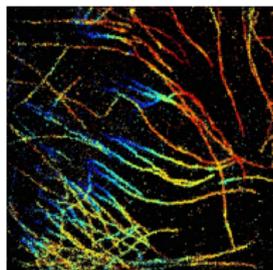


model selection via
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Beautiful mathematical core!



super-resolution microscopy



radar imaging

Structure of this talk

radar imaging
(main novelty)

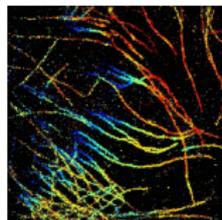


Structure of this talk

basic sparse signal recovery

$$\mathbf{y} = \mathbf{A} \mathbf{x}$$

super-resolution microscopy



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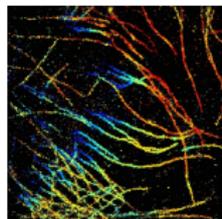
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A has random
uncorrelated columns

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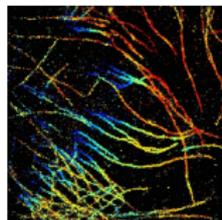
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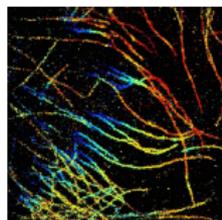
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Basic theory of sparse signal recovery

E. Candès
D. Donoho
J. Romberg
T. Tao

...

Prototypical model

$$\begin{array}{|c} y \end{array} = \begin{array}{|c|c|c|} \hline & & \\ \hline A & & \\ \hline \end{array} \begin{array}{|c} x \end{array}$$

Assumptions:

- (1) y is M dimensional

Prototypical model

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Assumptions:

- (1) y is M dimensional
- (2) Sparsity: x has at most S nonzero entries ($S < M$)

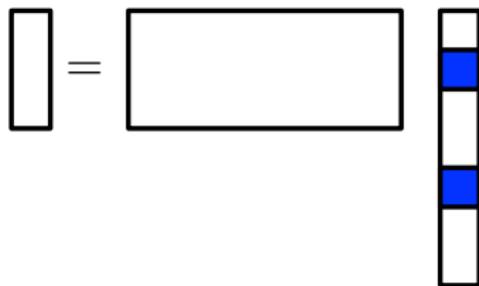
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$$\mathbf{y} = \mathbf{A} \mathbf{x}$$

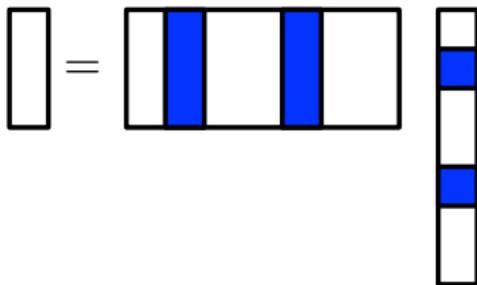
Assumptions:

- (1) \mathbf{y} is M dimensional
- (2) Sparsity: \mathbf{x} has at most S nonzero entries ($S < M$)
- (3) Properties of \mathbf{A} : $A_{ij} \sim \mathcal{N}(0, 1)$

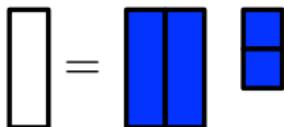
If we knew where nonzeros are ...



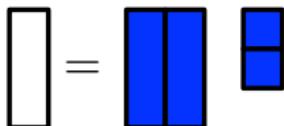
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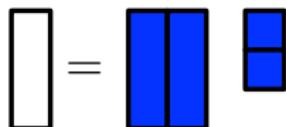


If we knew where nonzeros are ...



The search for nonzeros is combinatorial in nature!

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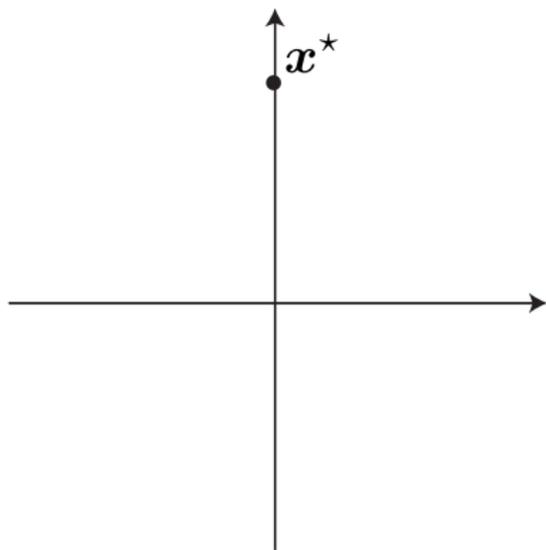
The search for nonzeros is combinatorial in nature!

Recovery by convex programming (relaxation):

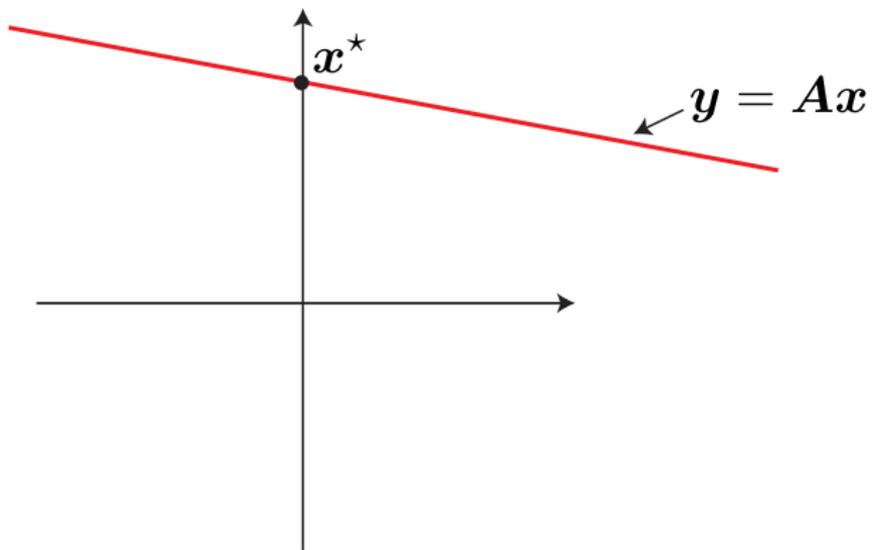
$$\text{minimize } \underbrace{\sum_i |x_i|}_{\text{l1-norm } \|\mathbf{x}\|_1} \quad \text{subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

Min norm problem is a convex program and computationally tractable

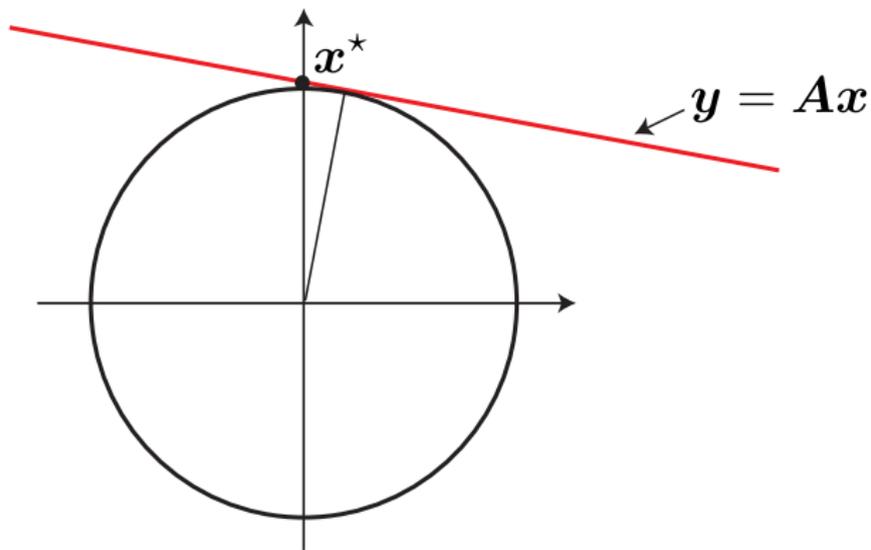
Why does ℓ_1 work?



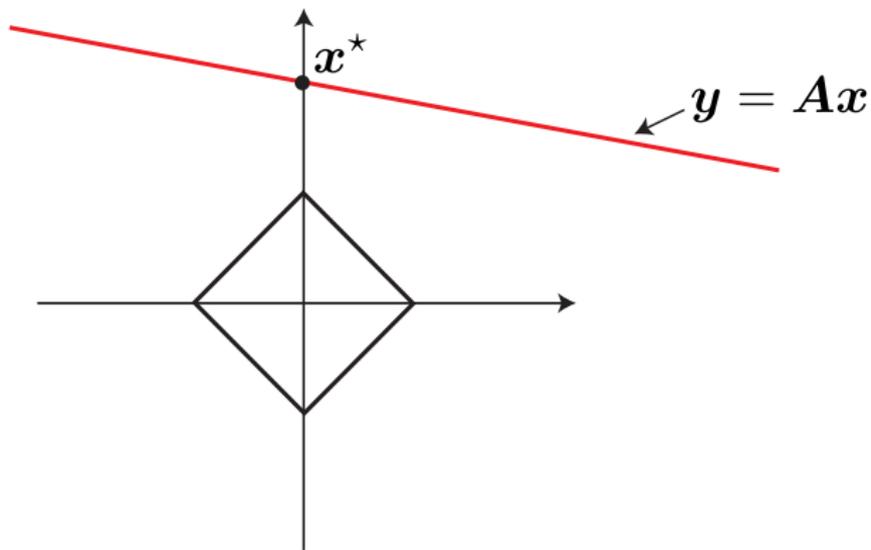
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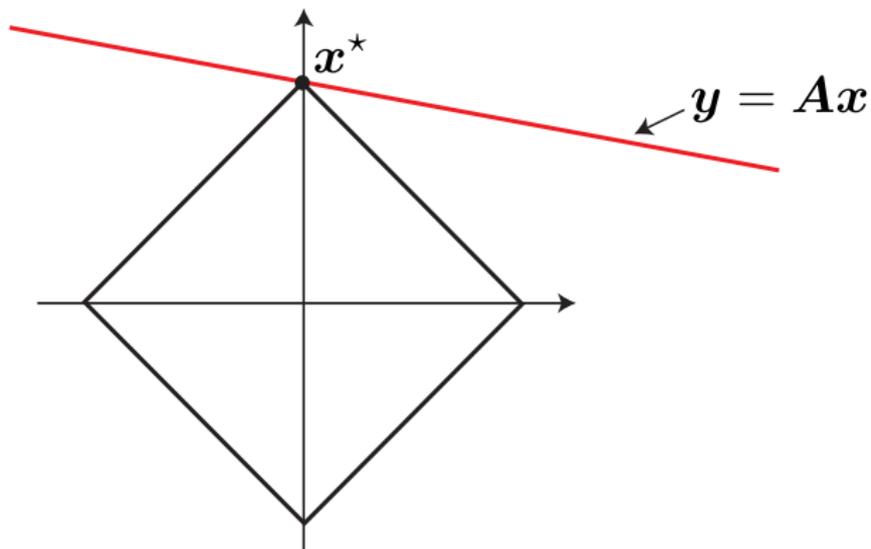
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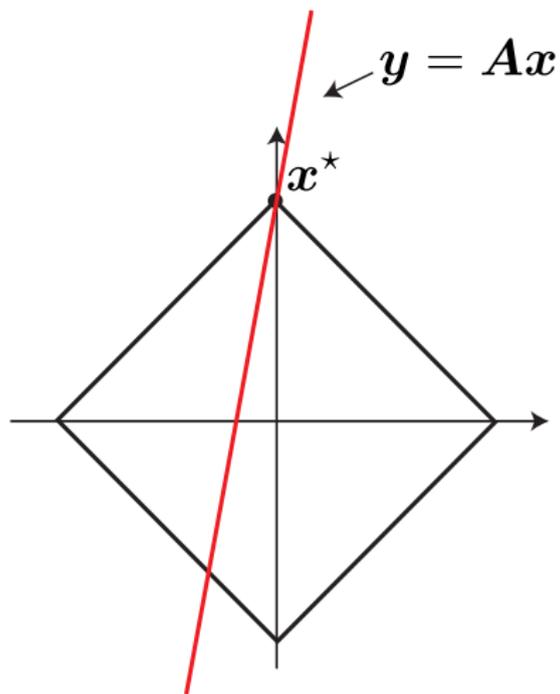
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Why does ℓ_1 work?



Why ℓ_1 may not always work

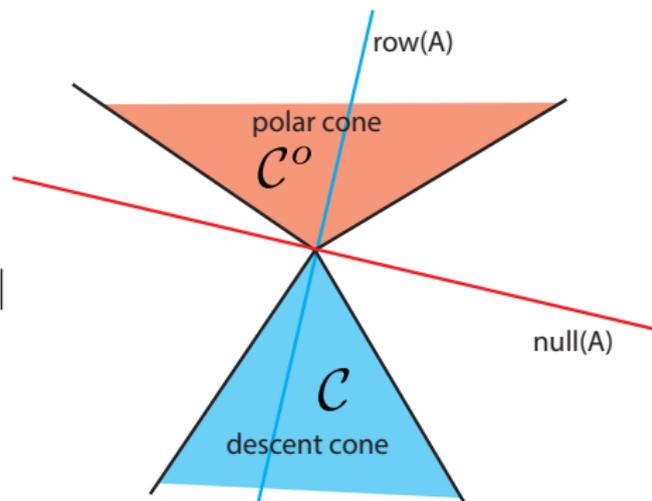


Dual certificates

$$\text{minimize } \|\mathbf{x}\|_1 \quad \text{such that } \mathbf{y} = \mathbf{A}\mathbf{x}$$

\mathbf{x}^* solution iff there exists

$$\mathbf{v} \perp \text{null}(\mathbf{A}) \text{ and } \mathbf{v} \in \mathcal{C}^o \Leftrightarrow \mathbf{v} \in \partial\|\mathbf{x}^*\|$$

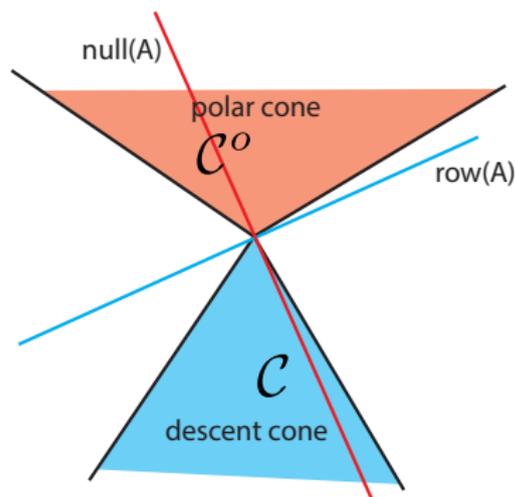


Dual certificates

$$\text{minimize } \|x\|_1 \quad \text{such that } y = Ax$$

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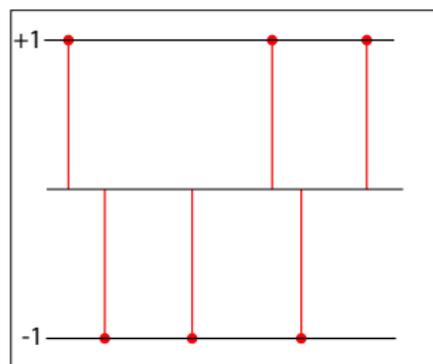


Construction of dual certificate

**dual
certificate**

$\mathbf{v} \in \text{row}(\mathbf{A})$ and

$$\begin{cases} v_i = \text{sgn}(x_i^*) & x_i^* \neq 0 \\ |v_i| < 1 & x_i^* = 0 \end{cases}$$

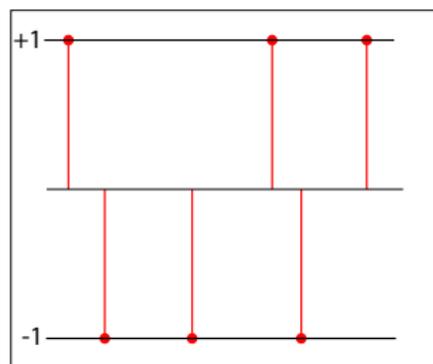


$\text{sgn}(\mathbf{x}^*)$

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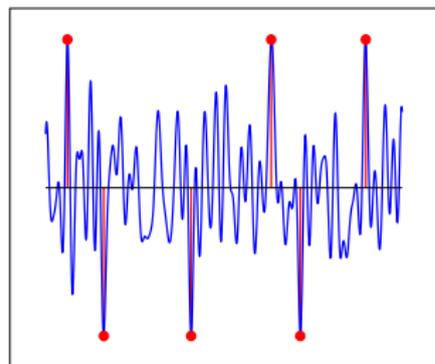


Least-squares solution to $\mathbf{v}_S = \text{sgn}(\mathbf{x})$:

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$\text{sgn}(\mathbf{x}^*)$

$$\mathbb{E} \mathbf{v}_{S^c} = \mathbf{0}$$

Least-squares solution to $\mathbf{v}_S = \text{sgn}(\mathbf{x})$:

$$\mathbf{A} = \begin{array}{|c|c|} \hline \mathbf{A}_S & \mathbf{A}_{S^c} \\ \hline \end{array}$$

off-support
support

$$\begin{array}{|c|} \hline \mathbf{v}_S \\ \hline \mathbf{v}_{S^c} \\ \hline \end{array} = \begin{array}{|c|} \hline \mathbf{A}_S^* \\ \hline \mathbf{A}_{S^c}^* \\ \hline \end{array} \mathbf{A}_S \left(\begin{array}{|c|} \hline \mathbf{A}_S^* \\ \hline \mathbf{A}_S \end{array} \right)^{-1} \begin{array}{|c|} \hline \mathbf{v}_S \\ \hline \end{array}$$

\uparrow
 $\text{sgn}(\mathbf{x}^*)$

Sparse recovery guarantee

$$\text{minimize } \|\mathbf{x}\|_1 \quad \text{subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

Assume:

- \mathbf{x} is arbitrary N -dimensional S -sparse vector
- data vector \mathbf{y} is M -dimensional with

$$M \geq S \log(N)$$

- $A_{ij} \sim \mathcal{N}(0, 1)$

Then, with high probability, l1 solution is exact!

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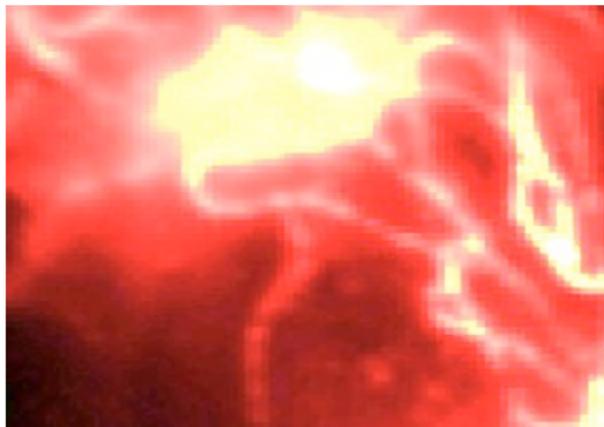
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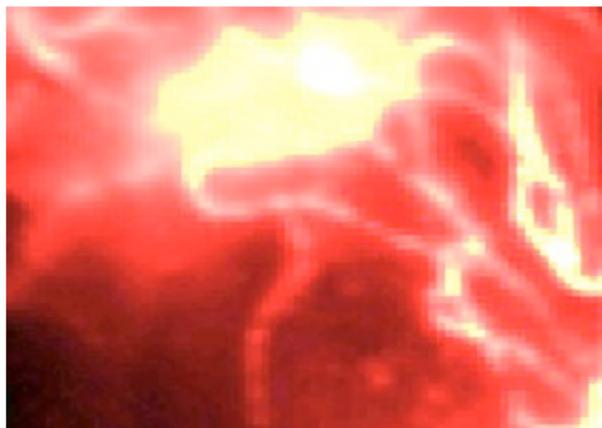
The **log** is needed to bound deviations of \mathbf{v}_{S^c} around $\mathbb{E} \mathbf{v}_{S^c} = \mathbf{0}$.

Super-resolution microscopy

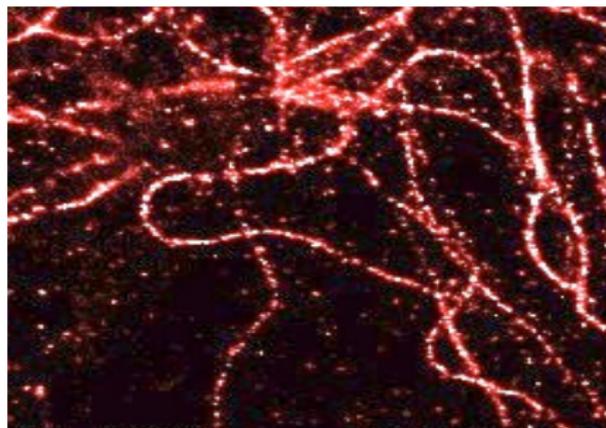
Abbe's diffraction limit for microscopy



Nobel Prize in Chemistry 2014

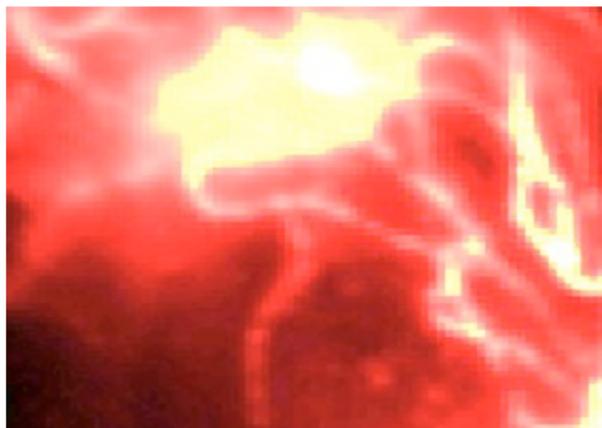


conventional microscopy

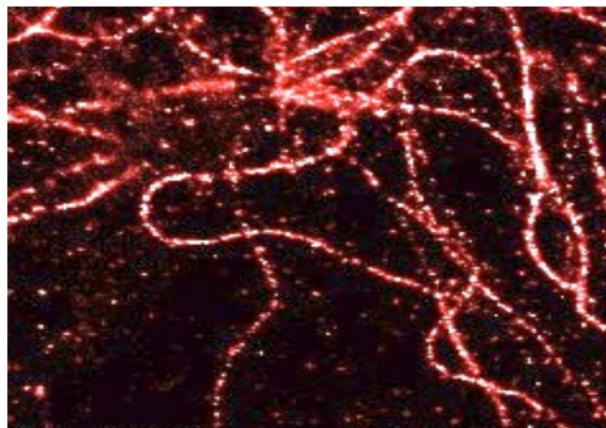


single-molecule microscopy

Nobel Prize in Chemistry 2014



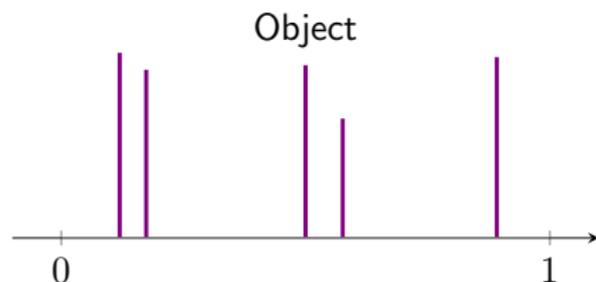
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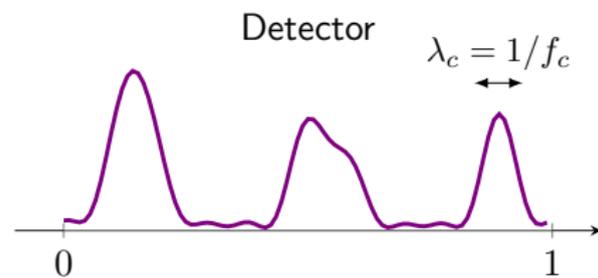
single-molecule microscopy

To make imaging faster, need a powerful algorithm for sparse signal recovery problem!

Mathematical model

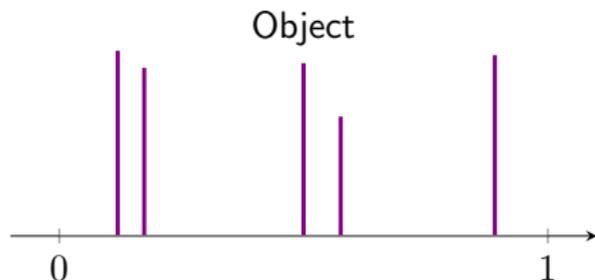


$$x(t) = \sum_s x_s \delta(t - t_s)$$

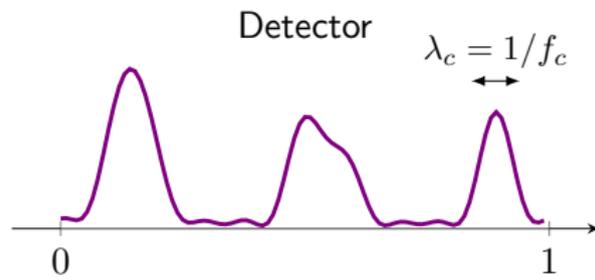


$$\begin{aligned} s(t) &= (f_{\text{low}} \star x)(t) \\ &= \sum_s x_s f_{\text{low}}(t - t_s) \end{aligned}$$

Mathematical model



$$x(t) = \sum_s x_s \delta(t - t_s)$$



$$\begin{aligned} s(t) &= (f_{\text{low}} \star x)(t) \\ &= \sum_s x_s f_{\text{low}}(t - t_s) \end{aligned}$$

$$\mathbf{x} = [x_0 \cdots x_{N-1}]^T$$

\mathbf{x} is **sparse**

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

\mathbf{A} ... $2f_c \times N$ low-frequency DFT

$$A_{kt} = e^{-i2\pi kt/N}, \quad |k| \leq f_c$$

Columns of \mathbf{A} are highly correlated

Solve:

minimize $\|\mathbf{x}\|_1$ subject to $\mathbf{y} = \mathbf{A}\mathbf{x}$

Question:

When does l1 work?

Columns of \mathbf{A} are highly correlated

Solve:

$$\text{minimize } \|\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

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When does l1 work?

First observation:

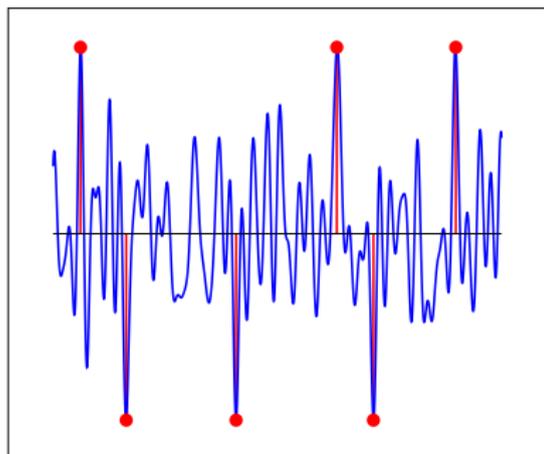
$$\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_N] \quad 2f_c \times N$$

$$A_{kt} \dots \text{ Gaussian:} \\ \langle \mathbf{a}_l, \mathbf{a}_{l+1} \rangle \approx \frac{1}{\sqrt{2f_c}}$$

$$A_{kt} \dots e^{i2\pi kt}, |k| < f_c \\ \langle \mathbf{a}_l, \mathbf{a}_{l+1} \rangle \approx 1$$

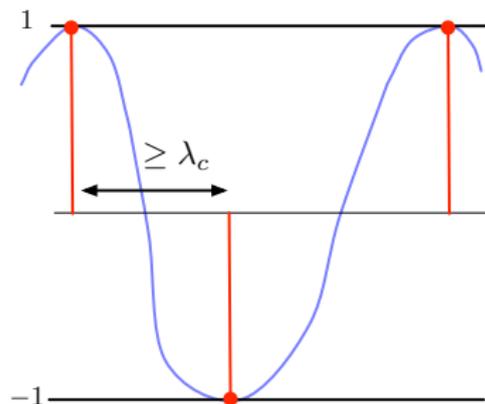
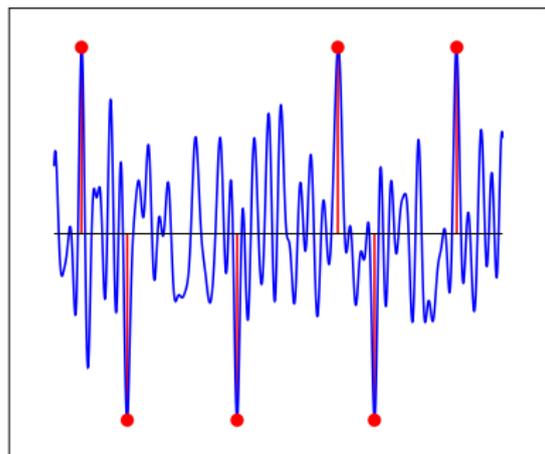
The dual polynomial for super-resolution

$$A_{kt} = e^{-i2\pi kt/N}, |k| \leq f_c \Rightarrow v(t) = \sum_{m=-f_c}^{f_c} \hat{v}_m e^{i2\pi mt}$$



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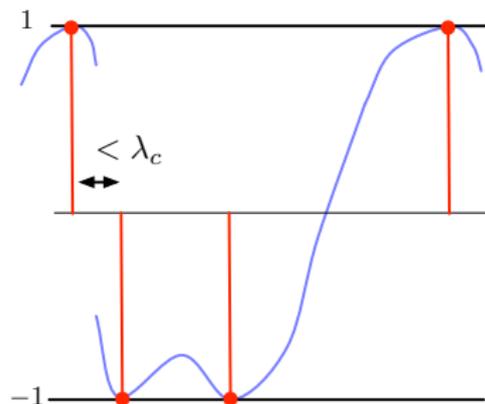
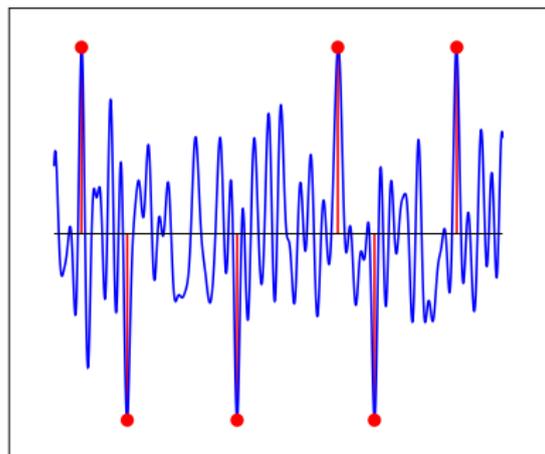


E. Candès and C. Fernandez-Granda '14

L1 works if spikes are further than $2\lambda_c$

The dual polynomial for super-resolution

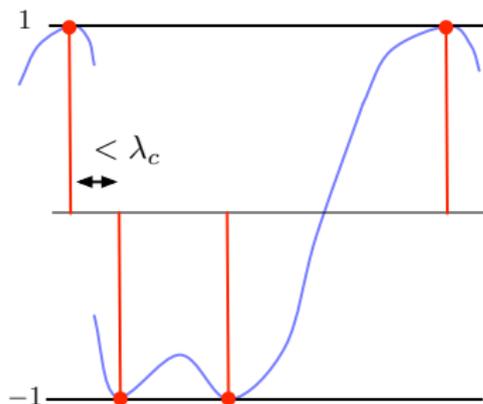
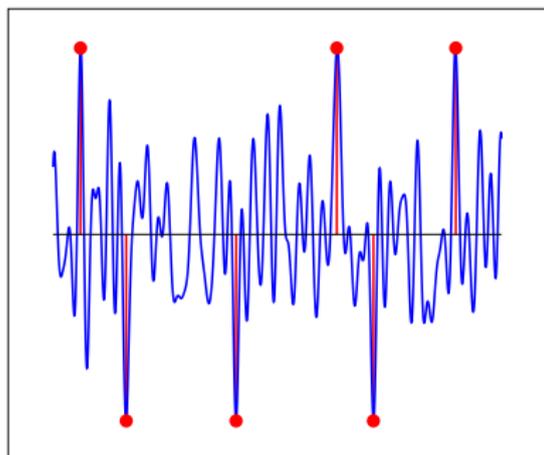
$$A_{kt} = e^{-i2\pi kt/N}, |k| \leq f_c \Rightarrow v(t) = \sum_{m=-f_c}^{f_c} \hat{v}_m e^{i2\pi mt}$$



If spikes are closer than $\lambda_c \Rightarrow$ L1 breaks

The dual polynomial for super-resolution

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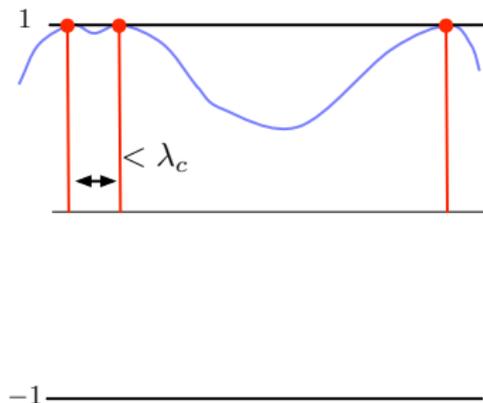
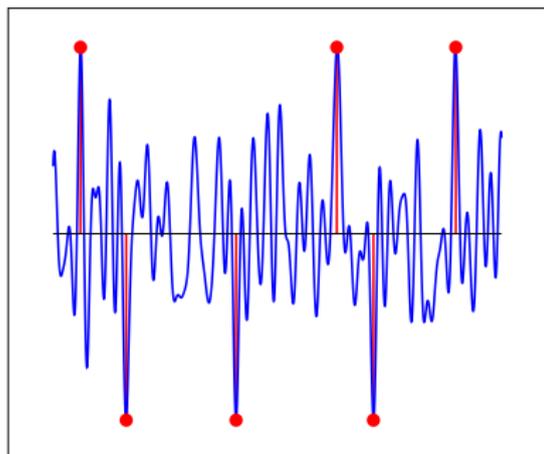
Bernstein theorem:

Consider: $v(t) = \sum_{k=-f_c}^{f_c} \hat{v}_k e^{-i2\pi kt}$ with $|v(t)| \leq 1$ for all t

Then: $|v'(t)| \leq 2f_c$ for all t .

The dual polynomial for super-resolution

$$A_{kt} = e^{-i2\pi kt/N}, |k| \leq f_c \Rightarrow v(t) = \sum_{m=-f_c}^{f_c} \hat{v}_m e^{i2\pi mt}$$



Donoho '92:

$\mathbf{x} \geq \mathbf{0} \Rightarrow$ L1 works if the number of spikes is less than $f_c + 1$

Super-resolution in the presence of noise

Model:

$$\mathbf{s} = f_{\text{low}} \star \hat{\mathbf{x}} + \mathbf{z}, \quad \|\mathbf{z}\|_1 \leq \delta$$

Solve:

$$\text{minimize } \|\mathbf{s} - f_{\text{low}} \star \hat{\mathbf{x}}\|_1 \quad \text{subject to } \hat{\mathbf{x}} \geq 0$$

Theorem: [V. Morgenshtern and E. Candès, 2015]

Assume $\mathbf{x} \geq 0$, \mathbf{x} is r -regular. Then,

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_1 \leq \delta \left(\frac{N}{2f_c} \right)^{2r}.$$

Super-resolution in the presence of noise

Model:

$$\mathbf{s} = f_{\text{low}} \star \hat{\mathbf{x}} + \mathbf{z}, \quad \|\mathbf{z}\|_1 \leq \delta$$

Solve:

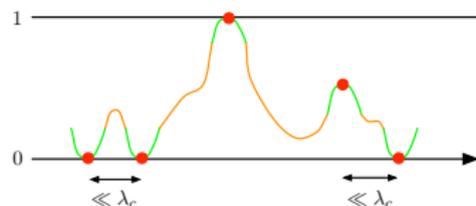
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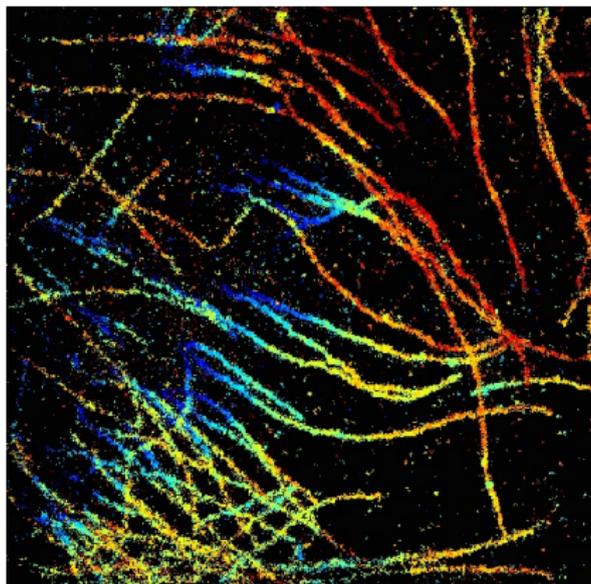
$$\|\hat{\mathbf{x}} - \mathbf{x}\|_1 \leq \delta \left(\frac{N}{2f_c} \right)^{2r}.$$

Key novelty: a set of new tools in Fourier analysis



Reconstruction of 3D signals from 2D data

Preliminary result: **4 times faster than state-of-the-art**



10000 CVX problems solved
TFOCS first order solver
millions of variables

Radar imaging

Radar imaging

Recap:

- Dual certificate is a tool to analyse success of l1.
- Structure of \mathbf{A} determines when certificate exists/does not exist.
- When A_{kl} are i.i.d. Gaussian, dual certificate is random. It exists if there are sufficiently many measurements.
- In the super-resolution problem, the certificate is deterministic low frequency trigonometric polynomial. It exists if the spikes are sufficiently separated.

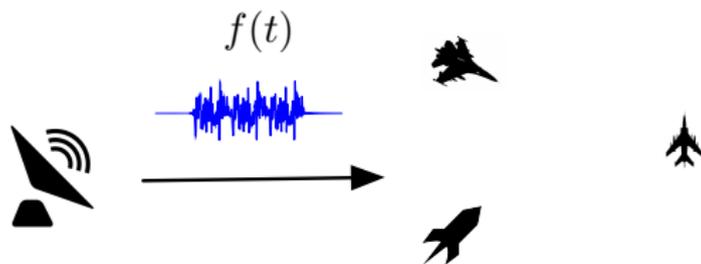
Radar imaging

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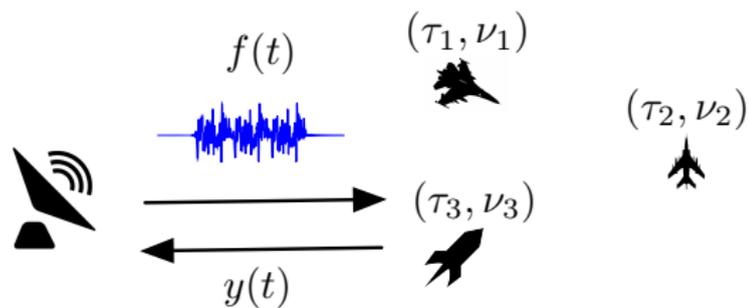
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We will see: in radar, the certificate is random **and** it approximates a low frequency trigonometric polynomial.

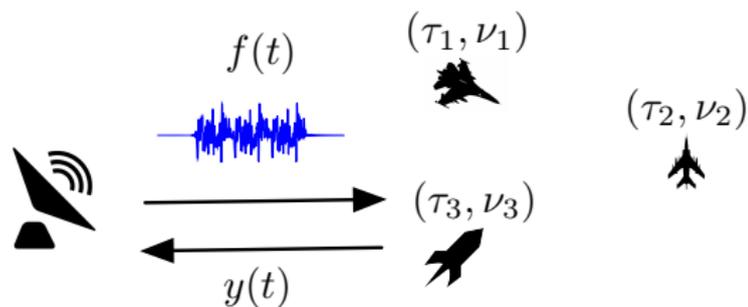
Mathematical model



Mathematical model

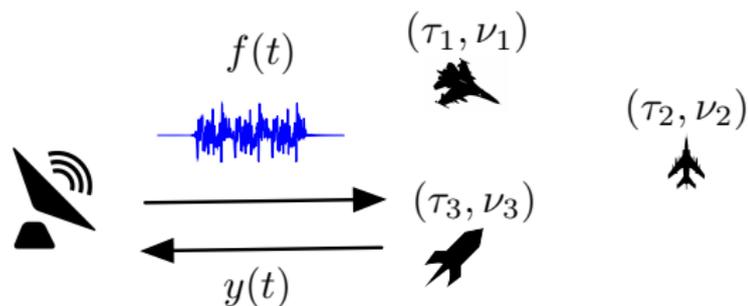


Mathematical model



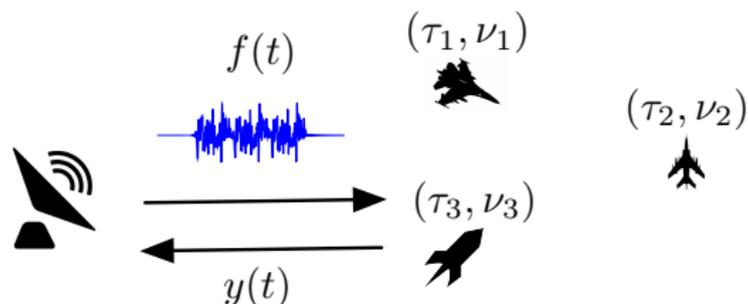
$$y(t) = \sum_{s=1}^S x_s(\mathcal{T}_{\tau_s} \mathcal{F}_{\nu_s} f)(t)$$

Mathematical model



$$\begin{aligned} y(t) &= \sum_{s=1}^S x_s(\mathcal{T}_{\tau_s} \mathcal{F}_{\nu_s} f)(t) \\ &= \sum_{s=1}^S x_s f(t - \tau_s) e^{i2\pi\nu_s t} \end{aligned}$$

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Goal: recover (x_s, τ_s, ν_s)

Time and bandwidth limitations

In practice:

- $f(t)$ is bandlimited to B Hz
- $y(t)$ is observed over T sec
- $\Rightarrow y(t)$ is BT -dimensional

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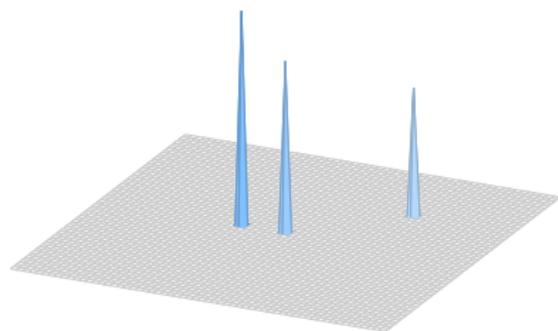
Goal: estimate τ and ν with precision higher than $1/B$ and $1/T$

$$y(t) = \sum_{s=1}^S x_s f(t - \tau_s) \quad (\text{super-resolution})$$

Blurring of time and frequency shifts

input

$$h(\tau, \nu) = \sum_s b_s \delta(\tau - \tau_s) \delta(\nu - \nu_s)$$

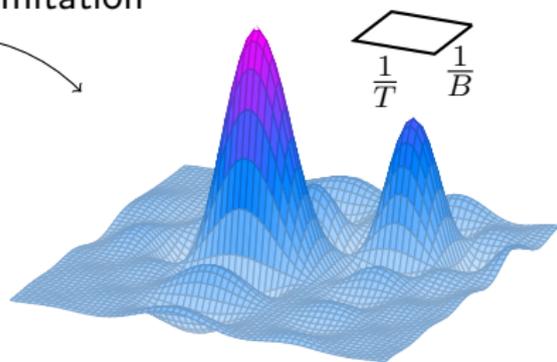
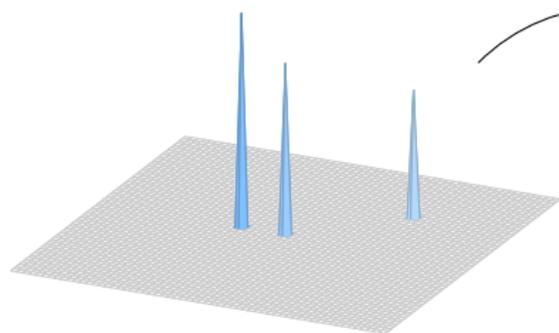


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band and time-limitation



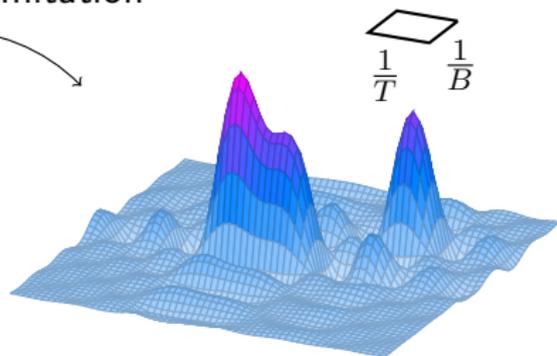
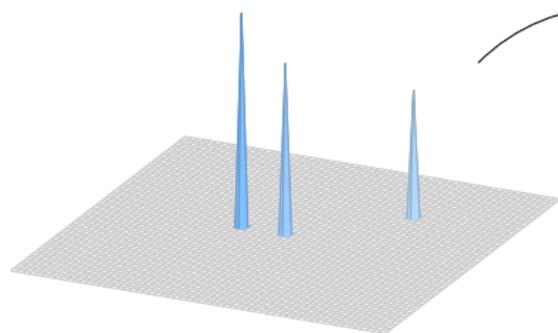
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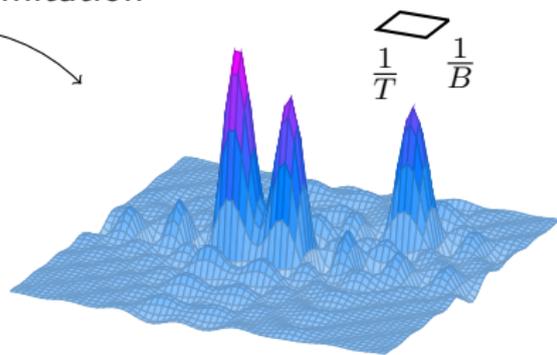
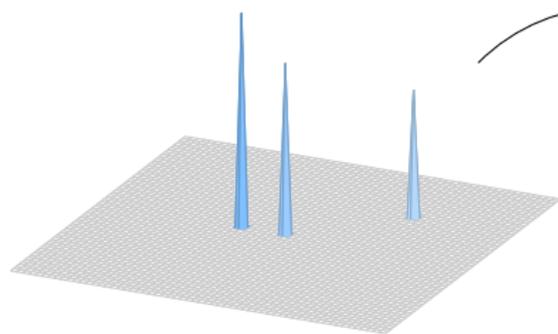
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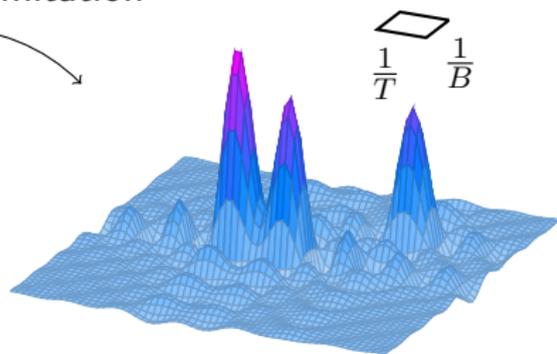
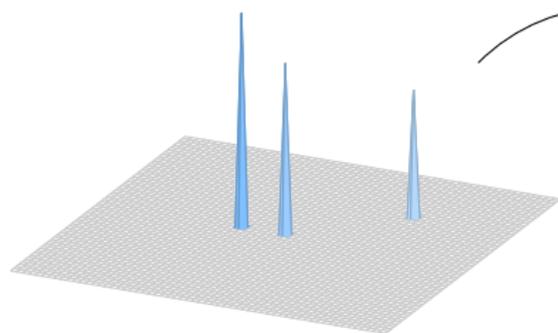
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$$y(t) = \iint h(\tau, \nu) f(t - \tau) e^{i2\pi\nu t} d\tau d\nu$$

band and time-limitation



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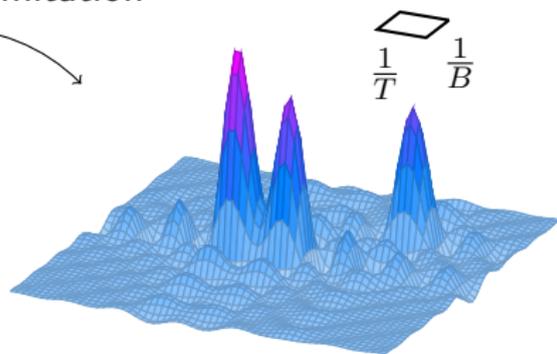
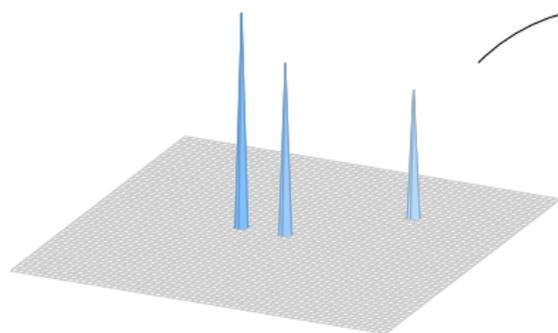
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$$\text{sinc}(\tau B) \text{sinc}(\nu T) * h(\tau, \nu)$$

Resolution achieved by classic Radar via matched filtering is $(\frac{1}{B}, \frac{1}{T})!$

Main result

Notation:

- \mathbf{y} , \mathbf{f} contain samples of $y(t)$, $f(t)$ at the rate $1/B$
- Columns of \mathbf{A} are $\mathcal{T}_\tau \mathcal{F}_\nu \mathbf{f}$ (indexed by τ and ν):

$$\mathbf{A} = \boxed{\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x}} \begin{matrix} \updownarrow \\ BT \end{matrix}$$

\longleftrightarrow
 $\gg (BT)^2$

- Random probing signal: f_ℓ i.i.d. $\mathcal{N}(0, 1)$
- \mathbf{x} contains x_s at location indexed by τ and ν

Solve:

$$\text{minimize } \|\mathbf{x}\|_1 \quad \text{subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

Main result

Solve:

$$\text{minimize } \|\mathbf{x}\|_1 \quad \text{subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

Theorem: [Heckel, Morgenshtern, Soltanolkotabi '15]

Assume:

$$|\tau_s - \tau_r| \geq \frac{5}{B} \quad \text{or} \quad |\nu_s - \nu_r| \geq \frac{5}{T}, \quad \text{for all } s \neq r$$

and

$$S \lesssim BT \log^{-3}(BT).$$

Then: with high probability, l1 minimization recovers \mathbf{x} exactly.

Hence, (τ_s, ν_s, x_s) are recovered perfectly.

Key novelty: dual polynomial for radar

Recall:

$$\mathbf{A} = \boxed{\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x}} \updownarrow BT$$

\longleftrightarrow

$$\gg (BT)^2$$

Need:

$$v(\tau, \nu) = [\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x}]^H \hat{\mathbf{v}}$$

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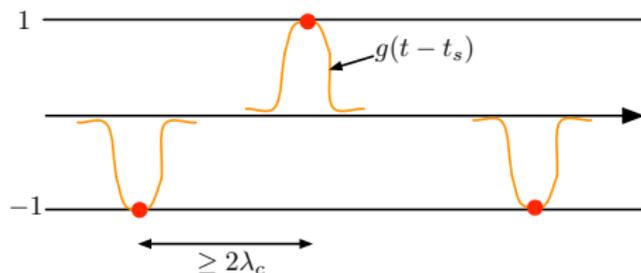
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Ingredient: dual certificate for super-resolution (with separation)
[Candes and Fernandez-Granda '14]

$$v(t) = \sum_s c_s g(t - t_s) + \text{corrections}$$

Low pass and concentrated kernel: $g(t) = \sum_{k=-f_c}^{f_c} \hat{g}_k e^{i2\pi kt}$



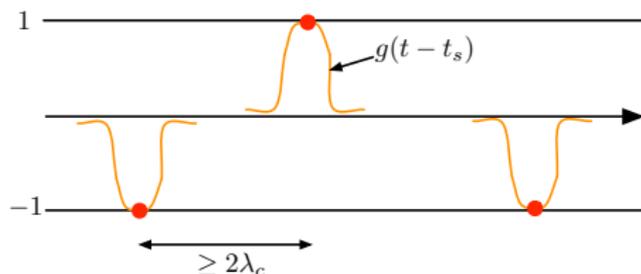
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For radar:

$$v(\tau, \nu) = \sum_s c_s g_{\tau_s, \nu_s}(\tau, \nu) + \text{corrections}$$

$g_{\tau_s, \nu_s}(\tau, \nu)$ “resembles” $g(\cdot - \tau_s) \times g(\cdot - \nu_s)$

Key novelty: construction of $g_{\tau_s, \nu_s}(\cdot, \cdot)$

Kernel:

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Kernel:

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We can write:

$$(\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x})^H = \left[\dots e^{i2\pi(\tau r + \nu q)} \dots \right] \mathbf{F} \mathbf{G}^H$$

\mathbf{F} is 2D DFT matrix

$$\mathbf{G} = \boxed{\mathcal{F}_{\frac{r}{BT}} \mathcal{T}_{\frac{q}{BT}} \mathbf{x}} \begin{matrix} \updownarrow BT \\ \longleftrightarrow \\ (BT)^2 \end{matrix}$$

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Choose coefficients:

$$\hat{\mathbf{v}} = \mathbf{G} \mathbf{F}^H \left[\dots \hat{g}_r \hat{g}_q e^{-i2\pi(\tau r + \nu q)} \dots \right]^T$$

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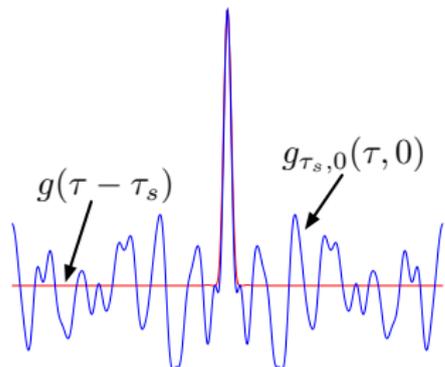
Observe: $\mathbb{E} [\mathbf{F} \mathbf{G}^H \mathbf{G} \mathbf{F}^H] = \mathbf{F} \mathbb{E} [\mathbf{G}^H \mathbf{G}] \mathbf{F}^H = \mathbf{I}$

Therefore: $\mathbb{E} [g_{\tau_s, 0}(\tau, 0)] = \sum_{k=-T}^T \hat{g}_k e^{i2\pi k(\tau - \tau_j)} = g(\tau - \tau_j)$

Random kernel approximates deterministic kernel

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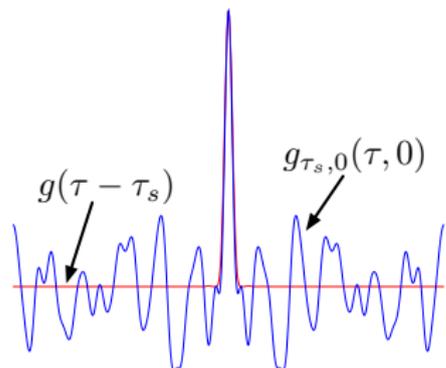
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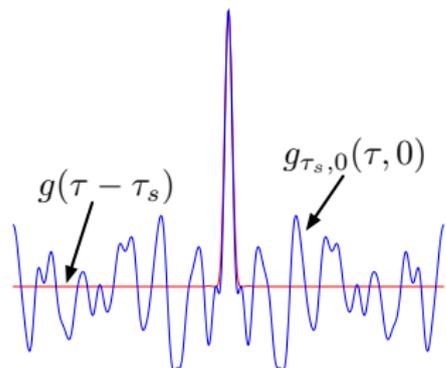
$$\begin{array}{c} \mathbf{v}_S \\ \dots \\ \mathbf{v}_{S^c} \end{array} = \begin{array}{c} \mathbf{A}_S^* \\ \dots \\ \mathbf{A}_{S^c}^* \end{array} \begin{array}{c} \mathbf{A}_S \\ \dots \\ \mathbf{A}_S \end{array} \left(\begin{array}{c} \mathbf{A}_S^* \\ \mathbf{A}_S \end{array} \right)^{-1} \begin{array}{c} \mathbf{1} \\ \uparrow \\ \text{sgn}(\mathbf{x}^*) \end{array}$$

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Random kernel approximates deterministic kernel

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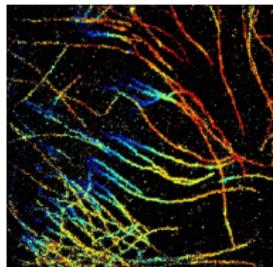
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Now we can use:

$$v(\tau, \nu) = \sum_S c_S g_{\tau_S, \nu_S}(\tau, \nu) + \text{corrections}$$

Mathematics of information

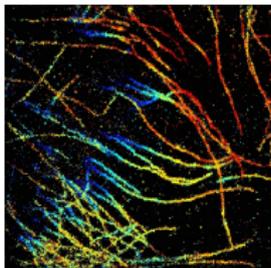


super-resolution microscopy



radar imaging

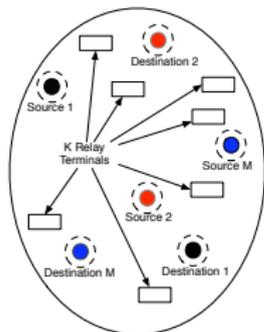
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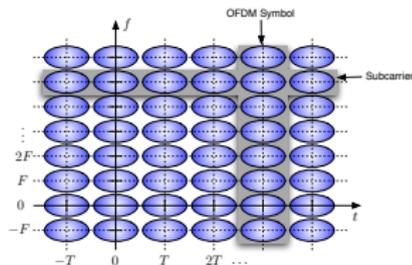
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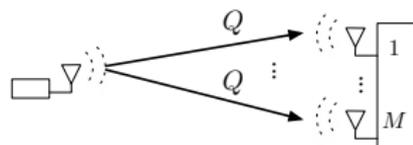
radar imaging



large wireless networks



communication under channel uncertainty



systems with multiple receive antennas

Related open problems

Phase retrieval from Fourier data:

$$y_k = |\langle f_k | x \rangle|^2$$

2D DFT (oversampled $\times 2$)

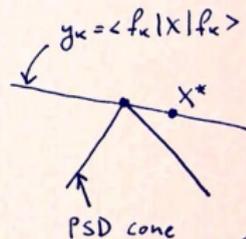
Applications:

- imaging of macromolecules
- astronomy
- speech processing

Iterative projection algorithms (Fienup, Gerchberg-Saxton)
often find approx. solutions ...
Unclear why!

Lifting: $y_k = \langle f_k | \underbrace{x}_{\underline{X}} | f_k \rangle^*$

SDP:
$$\begin{cases} \text{find } \underline{X} \\ \text{s.t. } y_k = \langle f_k | \underline{X} | f_k \rangle \\ \underline{X} \succeq 0, \text{rank}(\underline{X}) = 1 \end{cases}$$



very close, but not
(always) exact...

Thank you