Solving Underdetermined Linear Systems with Highly Correlated Columns

Veniamin I. Morgenshtern

Statistics Department, Stanford

"Big data" world

Problem: Estimate a **high-dimensional** object from (relatively) **few** corrupted data points.

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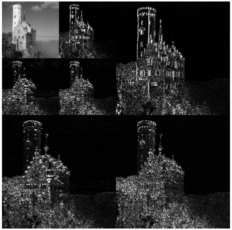

signal processing

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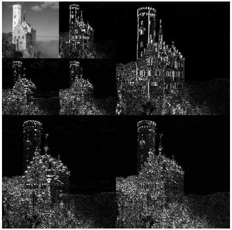

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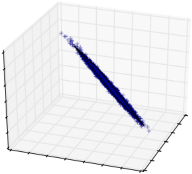

statistics

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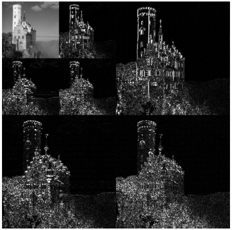

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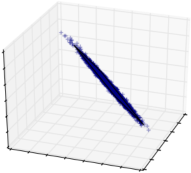

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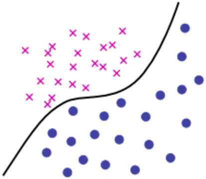
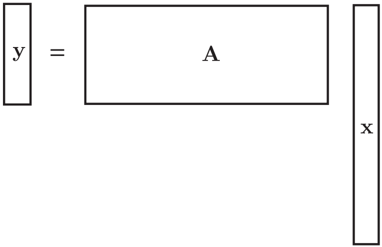


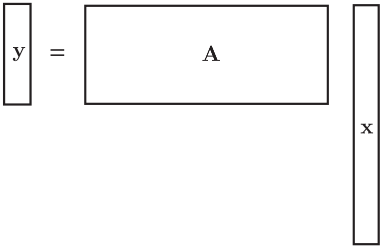
machine learning

Underdetermined linear system



Few linear measurements about a high-dimensional object

Underdetermined linear system



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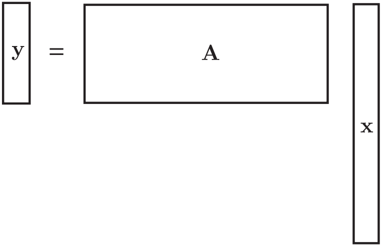

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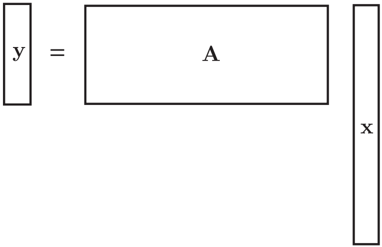

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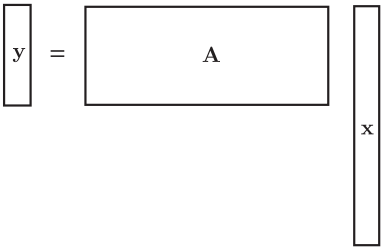

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How can we possibly recover the object?

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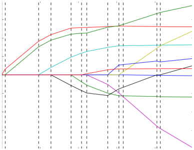
Many problems are naturally of this form.

Even more problems can be forced into this form!

Huge range of **“big data”** applications

compressed sensing
MRI



model selection via
LASSO

low rank matrix
recovery
(Netflix Prize)

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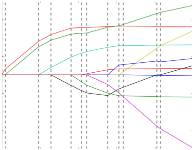

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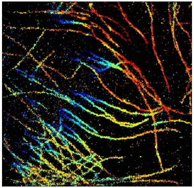

super-resolution microscopy

radar imaging

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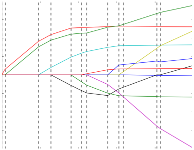

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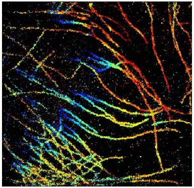
Beautiful mathematical core!



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radar imaging

Structure of this talk

radar imaging
(main novelty)

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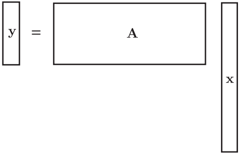
basic sparse signal recovery



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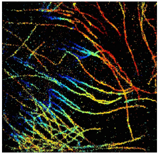

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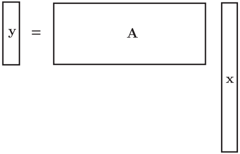
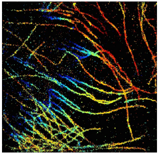


$\mathbf{A}$ has random uncorrelated columns

super-resolution microscopy



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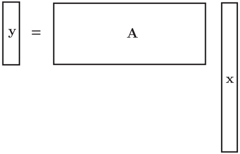

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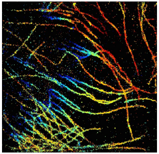

$\mathbf{A}$ has deterministic highly correlated columns

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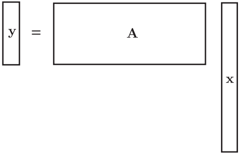

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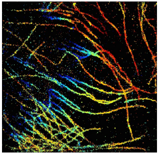

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radar imaging
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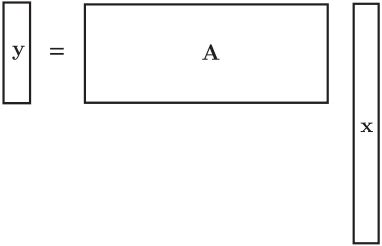
Basic theory of sparse signal recovery

E. Candès
D. Donoho
J. Romberg
T. Tao
...

Prototypical model



Assumptions:

(1) $\mathbf{y}$ is M dimenstional

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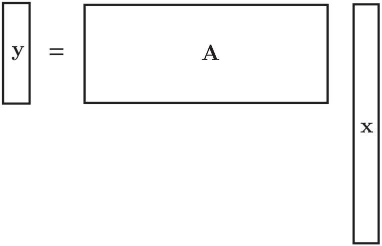

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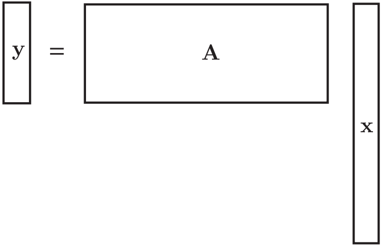

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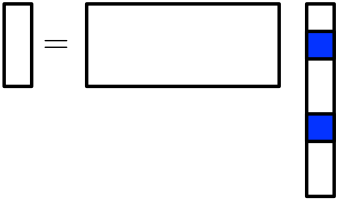
(3) Properties of $\mathbf{A}$: $A_{ij} \sim \mathcal{N}(0, 1)$

If we knew where nonzeros are ...



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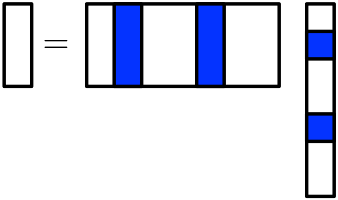

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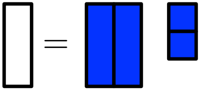

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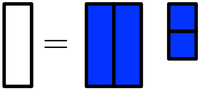

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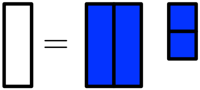


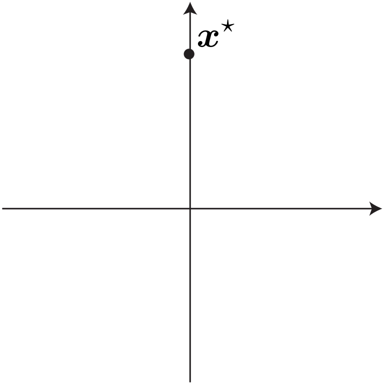
The search for nonzeros is combinatorial in nature!

Recovery by convex programming (relaxation):

$$\text{minimize} \quad \underbrace{\sum_i |x_i|}_{\text{l1-norm } \|\mathbf{x}\|_1} \quad \text{subject to} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$$

Min norm problem is a convex program and computationally tractable

Why does ℓ_1 work?



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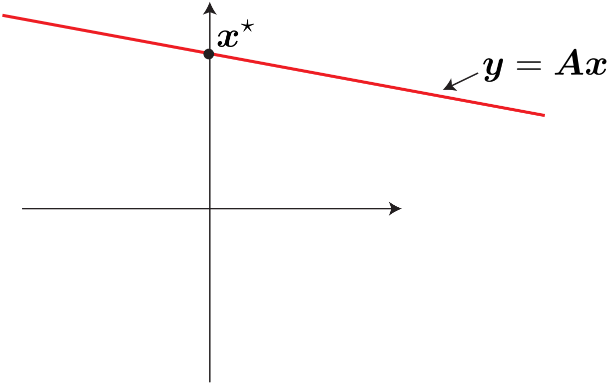

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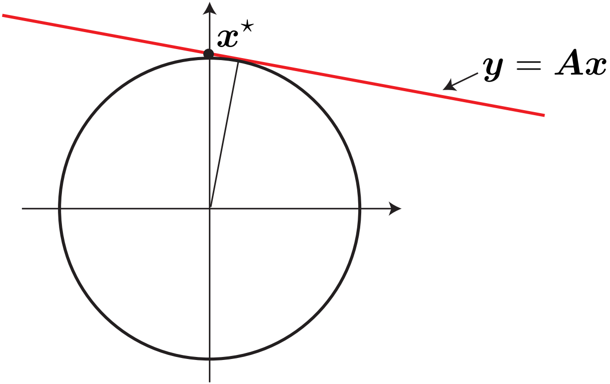

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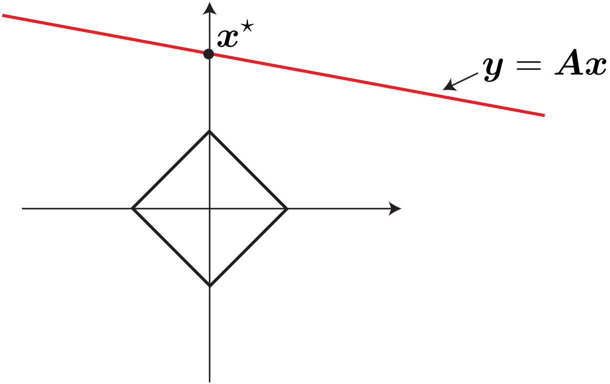

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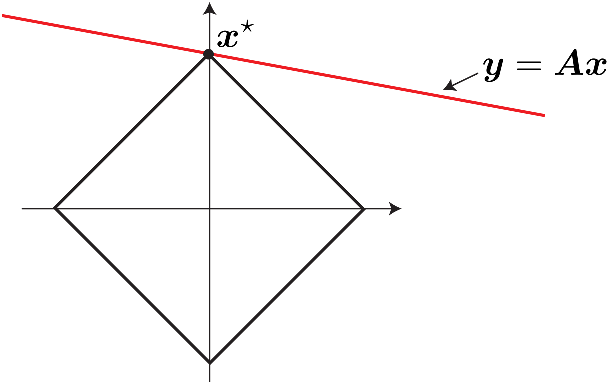
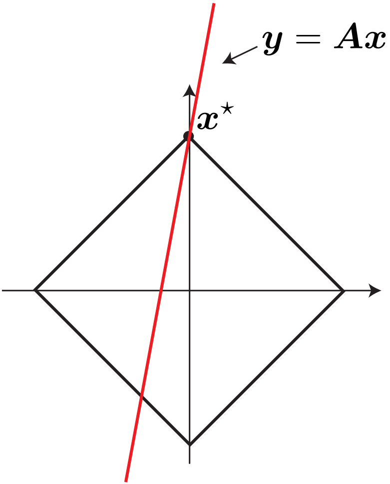


Why ℓ_1 may not always work

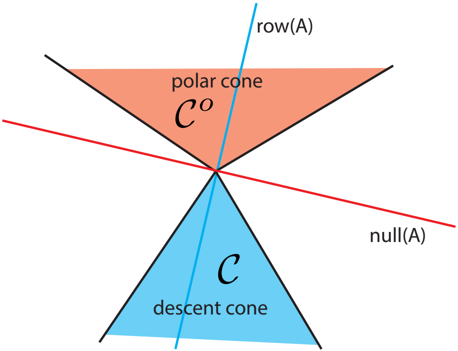


Dual certificates

$$\text{minimize} \quad \|\mathbf{x}\|_1 \quad \text{such that} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$$

$\mathbf{x}^*$ solution iff there exists

$\mathbf{v} \perp \text{null}(\mathbf{A})$ and $\mathbf{v} \in \mathcal{C}^o \Leftrightarrow \mathbf{v} \in \partial\|\mathbf{x}^*\|$



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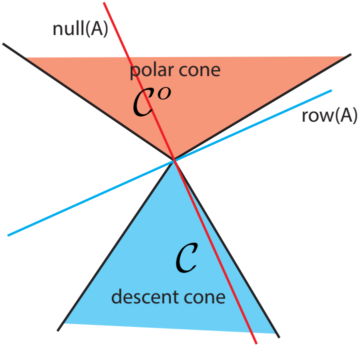
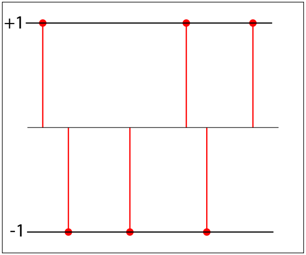


Construction of dual certificate

dual certificate $\quad \mathbf{v} \in \mathrm{row}(\mathbf{A}) \quad$ and $\quad \begin{cases} v_i = \mathrm{sgn}(x_i^*) & x_i^* \neq 0 \\ |v_i| < 1 & x_i^* = 0 \end{cases}$



$\mathrm{sgn}(\mathbf{x}^*)$

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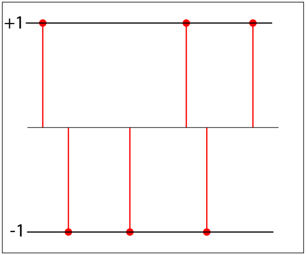

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Least-squares solution to $\mathbf{v}_S = \text{sgn}(\mathbf{x})$:

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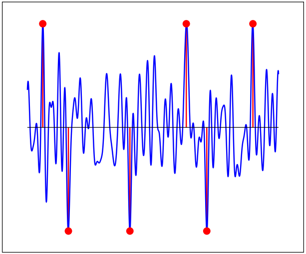

$\text{sgn}(\mathbf{x}^*)$

$\mathbb{E}\, \mathbf{v}_{S^c} = \mathbf{0}$

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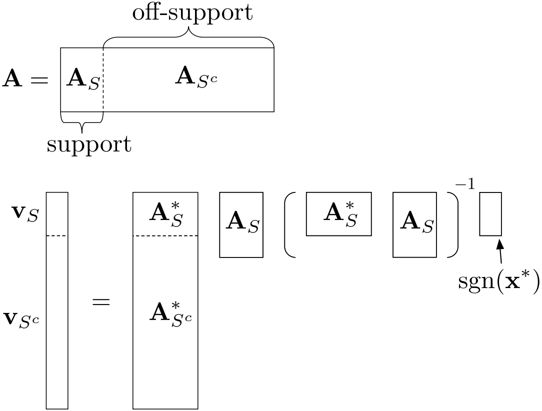

Sparse recovery guarantee

$$\text{minimize } \|\mathbf{x}\|_1 \ \text{ subject to } \ \mathbf{y} = \mathbf{A}\mathbf{x}$$

Assume:

- $\mathbf{x}$ is arbitrary N-dimensional S-sparse vector
- data vector $\mathbf{y}$ is M-dimensional with

$$M \geq S \log(N)$$

- $A_{ij} \sim \mathcal{N}(0, 1)$

Then, with high probability, l1 solution is exact!

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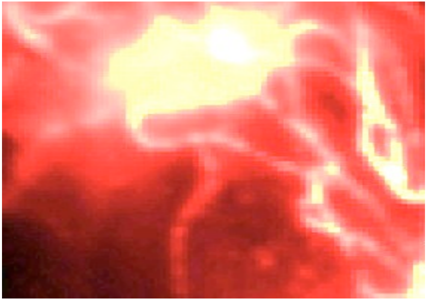
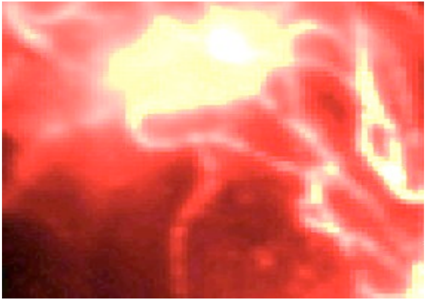
The **log** is needed to bound deviations of $\mathbf{v}_{S^c}$ around $\mathbb{E}\,\mathbf{v}_{S^c} = \mathbf{0}$.

Super-resolution microscopy

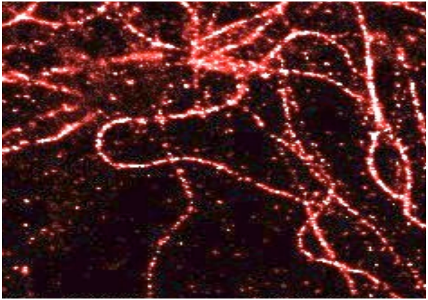
Abbe's diffraction limit for microscopy



Nobel Prize in Chemistry 2014



conventional microscopy



single-molecule microscopy

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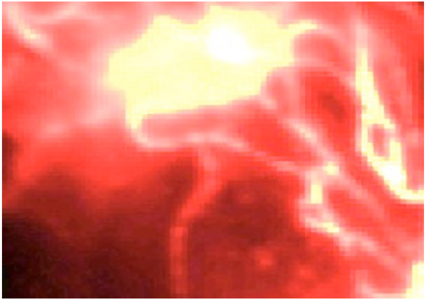

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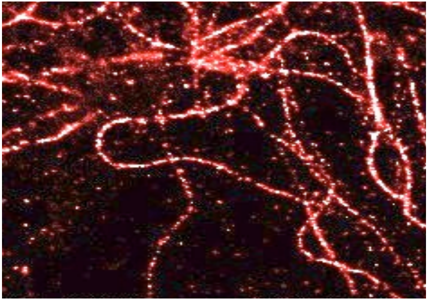
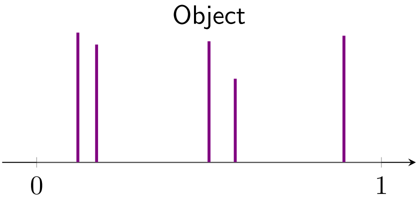
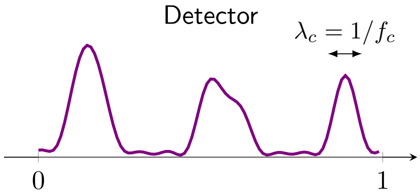


single-molecule microscopy

To make imaging faster, need a powerfull algorithm for sparse signal recovery problem!

Mathematical model



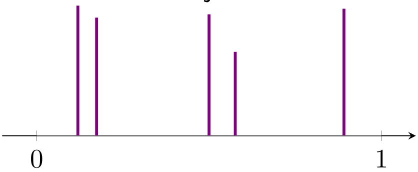


$$x(t) = \sum_s x_s \delta(t - t_s)$$

$$s(t) = (f_{\text{low}} \star x)(t) = \sum_s x_s f_{\text{low}}(t - t_s)$$

Mathematical model

Object



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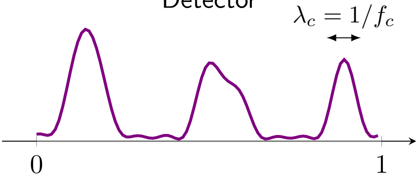
Detector



$$s(t) = (f_{\text{low}} \star x)(t) = \sum_s x_s f_{\text{low}}(t - t_s)$$

$$\mathbf{x} = [x_0 \cdots x_{N-1}]^{\mathsf{T}}$$

$\mathbf{x}$ is **sparse**

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$\mathbf{A}$ ... $2f_c \times N$ low-frequency DFT

$$A_{kt} = e^{-i2\pi kt/N}, \ |k| \leq f_c$$

Columns of $\mathbf{A}$ are highly correlated

Solve:

minimize $\|\mathbf{x}\|_1$ subject to $\mathbf{y} = \mathbf{A}\mathbf{x}$

Question:

When does l1 work?

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Solve:

$$\text{minimize } \|\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x}$$

Question:

When does l1 work?

First observation:

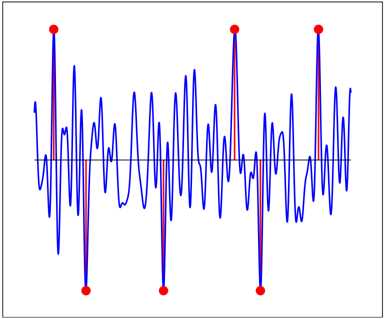
$$\mathbf{A} = [\mathbf{a}_1, \ldots \mathbf{a}_N] \quad 2f_c \times N$$

A_{kt} ... Gaussian:
$\langle \mathbf{a}_l, \mathbf{a}_{l+1} \rangle \approx \frac{1}{\sqrt{2f_c}}$

A_{kt} ... $e^{i2\pi kt}, |k| < f_c$
$\langle \mathbf{a}_l, \mathbf{a}_{l+1} \rangle \approx 1$

The dual polynomial for super-resolution

$$A_{kt} = e^{-i2\pi kt/N}, |k| \leq f_c \;\Rightarrow\; v(t) = \sum_{m=-f_c}^{f_c} \hat{v}_m e^{i2\pi mt}$$



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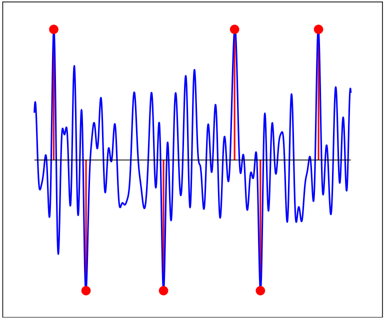

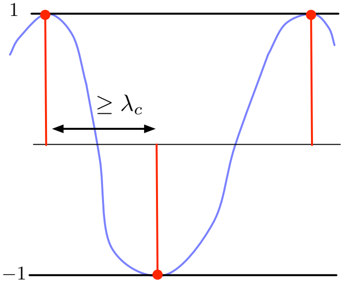

E. Candès and C. Fernandez-Granda '14

L1 works if spikes are further than $2\lambda_c$

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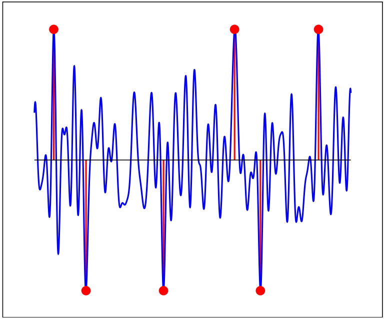

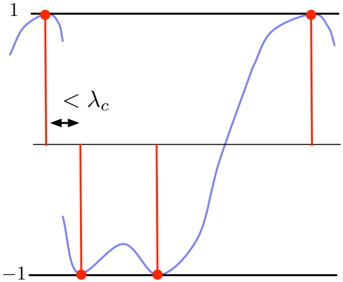

If spikes are closer than λ_c $\Rightarrow$ L1 breaks

The dual polynomial for super-resolution

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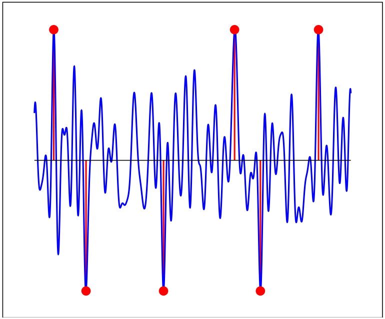

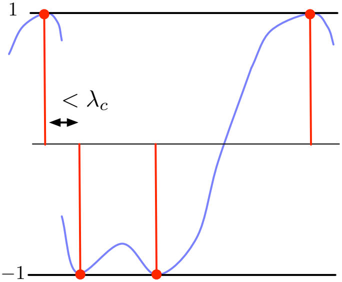

Bernstein theorem:

Consider: $v(t) = \sum_{k=-f_c}^{f_c} \hat{v}_k e^{-i2\pi kt}$ with $\left|v(t)\right| \le 1$ for all t

Then: $\left|v'(t)\right| \le 2f_c$ for all t.

The dual polynomial for super-resolution

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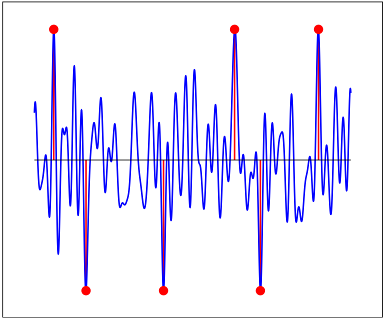

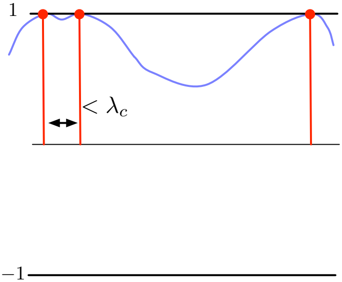

Donoho '92:

$\mathbf{x} \geq \mathbf{0} \;\Rightarrow$ L1 works if the number of spikes is less than $f_c + 1$

Super-resolution in the presence of noise

Model:

$$\mathbf{s} = f_{\text{low}} \star \hat{\mathbf{x}} + \mathbf{z}, \quad \|\mathbf{z}\|_1 \leq \delta$$

Solve:

$$\text{minimize } \|\mathbf{s} - f_{\text{low}} \star \hat{\mathbf{x}}\|_1 \quad \text{subject to} \quad \hat{\mathbf{x}} \geq 0$$

Theorem: [V. Morgenshtern and E. Candès, 2015]

Assume $\mathbf{x} \geq 0$, $\mathbf{x}$ is r-regular. Then,

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_1 \leq \delta \left(\frac{N}{2f_c} \right)^{2r}.$$

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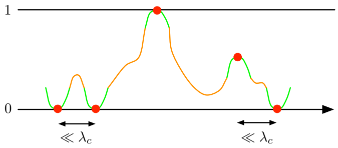
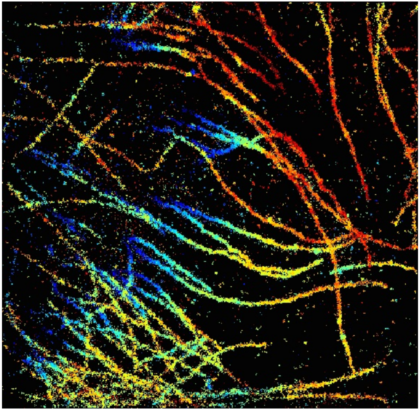
Key novelty: a set of new tools in Fourier analysis



Reconstruction of 3D signals from 2D data

Preliminary result: **4 times faster than state-of-the-art**



10000 CVX problems solved
TFOCS first order solver
millions of variables

Radar imaging

Radar imaging

Recap:

- Dual certificate is a tool to analyse success of l1.
- Structure of $\mathbf{A}$ determines when certificate exists/does not exist.
- When A_{kl} are i.i.d. Gaussian, dual certificate is random. It exists if there are sufficiently many measurements.
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Radar imaging

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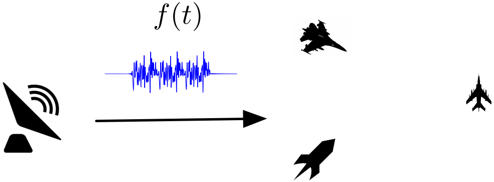
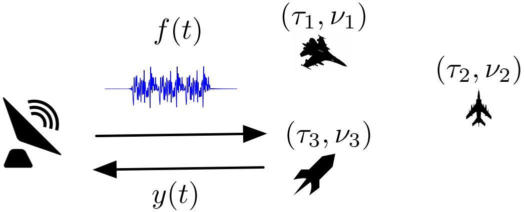
We will see: in radar, the certificate is random **and** it approximates a low frequency trigonometric polynomial.

Mathematical model



Mathematical model



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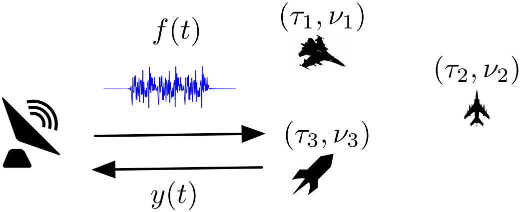
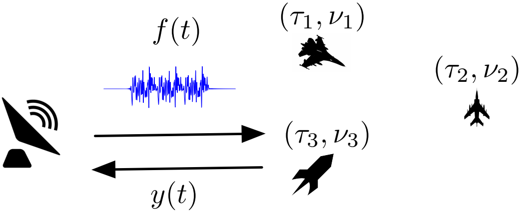


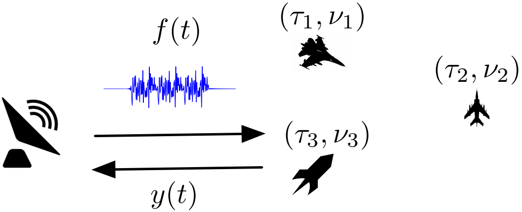
$$y(t) = \sum_{s=1}^{S} x_s (\mathcal{T}_{\tau_s} \mathcal{F}_{\nu_s} f)(t)$$

Mathematical model



$$
\begin{aligned}
y(t) &= \sum_{s=1}^{S} x_s (\mathcal{T}_{\tau_s} \mathcal{F}_{\nu_s} f)(t) \\
&= \sum_{s=1}^{S} x_s f(t - \tau_s) e^{i2\pi \nu_s t}
\end{aligned}
$$

Mathematical model



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$$= \sum_{s=1}^{S} x_s f(t - \tau_s) e^{i2\pi\nu_s t}$$

Goal: recover (x_s, τ_s, ν_s)

Time and bandwidth limitations

In practice:

- $f(t)$ is bandlimited to $B\,\mathrm{Hz}$
- $y(t)$ is observed over $T\,\mathrm{sec}$
- $\Rightarrow y(t)$ is BT-dimensional

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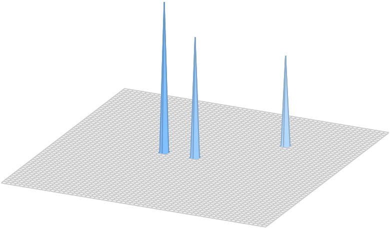
$$y(t) = \sum_{s=1}^{S} x_s f(t - \tau_s) \qquad \text{(super-resolution)}$$

Blurring of time and frequency shifts

input

$$h(\tau, \nu) = \sum_s b_s \delta(\tau - \tau_s) \delta(\nu - \nu_s)$$



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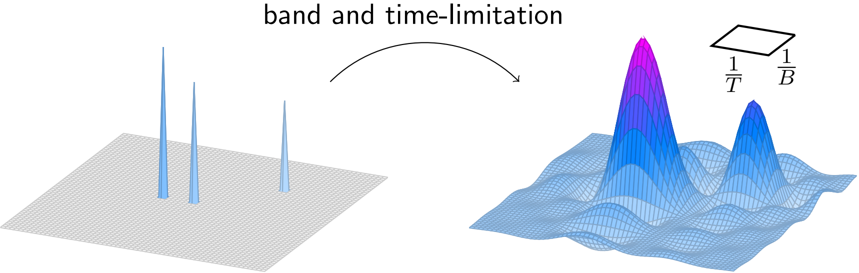

$$\operatorname{sinc}(\tau B)\operatorname{sinc}(\nu T) * h(\tau,\nu)$$

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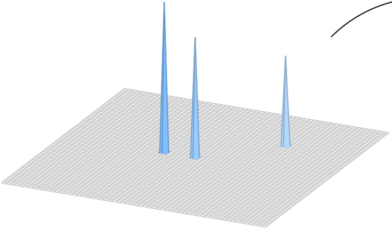
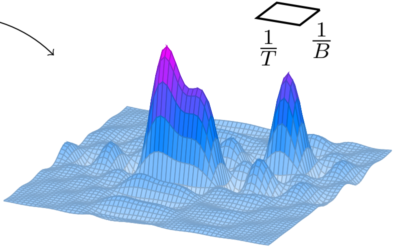
band and time-limitation





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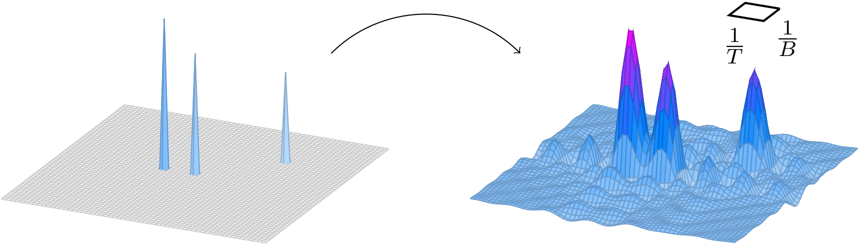

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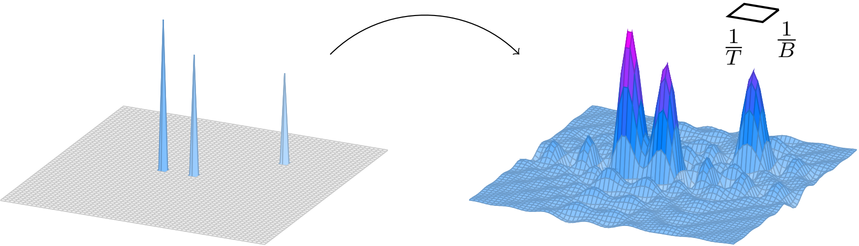

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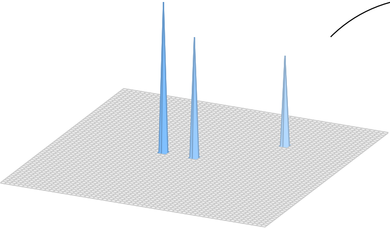

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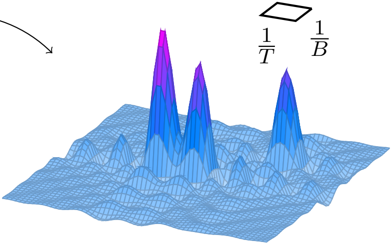

$$\mathrm{sinc}(\tau B)\,\mathrm{sinc}(\nu T) * h(\tau,\nu)$$

Resolution achieved by classic Radar via matched filtering is $\left(\frac{1}{B}, \frac{1}{T}\right)$!

Main result

Notation:

- $\mathbf{y}$, $\mathbf{f}$ contain samples of $y(t)$, $f(t)$ at the rate $1/B$
- Columns of $\mathbf{A}$ are $\mathcal{T}_\tau \mathcal{F}_\nu \mathbf{f}$ (indexed by τ and ν):

$$\mathbf{A} = \boxed{\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x}} \updownarrow BT$$

$$\longleftrightarrow$$

$$\gg (BT)^2$$

- Random probing signal: f_ℓ i.i.d. $\mathcal{N}(0,1)$
- $\mathbf{x}$ contains x_s at location indexed by τ and ν

Solve:

$$\text{minimize } \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$$

Main result

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Theorem: [Heckel, Morgenshtern, Soltanolkotabi '15]

Assume:

$$|\tau_s - \tau_r| \geq \frac{5}{B} \text{ or } |\nu_s - \nu_r| \geq \frac{5}{T}, \text{ for all } s \neq r$$

and

$$S \lesssim BT \log^{-3}(BT).$$

Then: with high probability, l1 minimization recovers $\mathbf{x}$ exactly.
Hence, (τ_s, ν_s, x_s) are recovered perfectly.

Key novelty: dual polynomial for radar

Recall:

$$\mathbf{A} = \boxed{\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x}} \updownarrow BT$$

$$\longleftrightarrow \gg (BT)^2$$

Need:

$$v(\tau, \nu) = [\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x}]^H \hat{\mathbf{v}}$$

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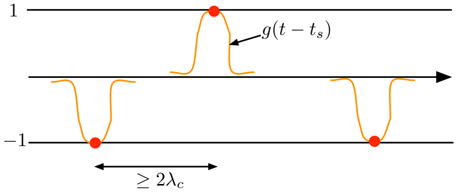
Ingredient: dual certificate for super-resolution (with separation) [Candes and Fernandez-Granda '14]

$$v(t) = \sum_s c_s g(t - t_s) + \text{corrections}$$

Low pass and concentrated kernel: $g(t) = \sum_{k=-f_c}^{f_c} \hat{g}_k e^{i2\pi kt}$



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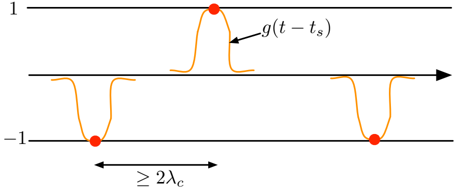

For radar:

$$v(\tau,\nu) = \sum_s c_s g_{\tau_s,\nu_s}(\tau,\nu) + \text{corrections}$$

$$g_{\tau_s,\nu_s}(\tau,\nu) \ \textbf{“resembles”} \ g(\cdot - \tau_s) \times g(\cdot - \nu_s)$$

Key novelty: construction of $g_{\tau_s,\nu_s}(\cdot,\cdot)$

Kernel:

$$g_{\tau_s,\nu_s}(\tau,\nu) = \left[\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x}\right]^H \hat{\mathbf{v}}$$

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Kernel:

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We can write:

$$(\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x})^{\mathsf{H}} = \left[\cdots e^{i2\pi(\tau r + \nu q)} \cdots\right] \mathbf{F}\mathbf{G}^{\mathsf{H}}$$

$\mathbf{F}$ is 2D DFT matrix

$$\mathbf{G} = \boxed{\mathcal{F}_{\frac{r}{BT}} \mathcal{T}_{\frac{q}{BT}} \mathbf{x}} \updownarrow BT$$

$$\longleftrightarrow$$
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$$\mathbf{G} = \boxed{\mathcal{F}_{\frac{r}{BT}} \mathcal{T}_{\frac{q}{BT}} \mathbf{x}} \updownarrow BT, \quad \longleftrightarrow (BT)^2$$

Choose coefficients:

$$\hat{\mathbf{v}} = \mathbf{G}\mathbf{F}^{\mathsf{H}} \left[\cdots \hat{g}_r \hat{g}_q e^{-i2\pi(\tau r + \nu q)} \cdots \right]^{\mathsf{T}}$$

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Observe: $\mathbb{E}\left[\mathbf{F}\mathbf{G}^{\mathsf{H}}\mathbf{G}\mathbf{F}^{\mathsf{H}}\right] = \mathbf{F}\,\mathbb{E}\left[\mathbf{G}^{\mathsf{H}}\mathbf{G}\right]\mathbf{F}^{\mathsf{H}} = \mathbf{I}$

Therefore: $\mathbb{E}\left[g_{\tau_s,0}(\tau,0)\right] = \sum_{k=-T}^{T} \hat{g}_k e^{i2\pi k(\tau-\tau_j)} = g(\tau-\tau_j)$

Random kernel approximates deterministic kernel

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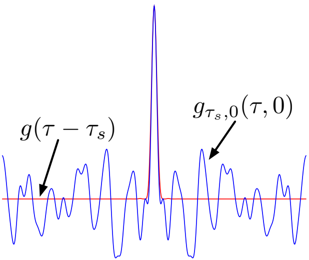

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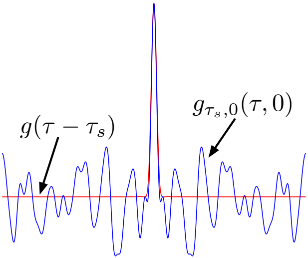

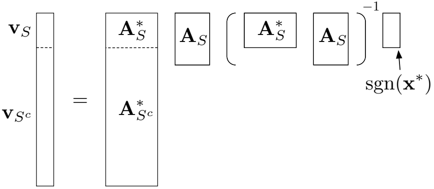

$$\mathbb{E}\,\mathbf{v}_{S^c} = \mathbf{0}$$

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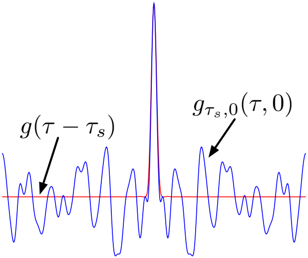
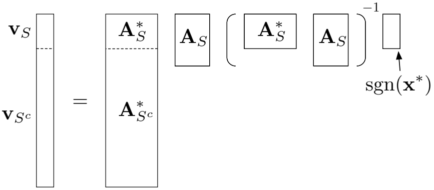


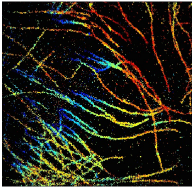


$$\mathbb{E}\,\mathbf{v}_{S^c} = \mathbf{0}$$

Now we can use:

$$v(\tau,\nu) = \sum_s c_s g_{\tau_s,\nu_s}(\tau,\nu) + \text{corrections}$$

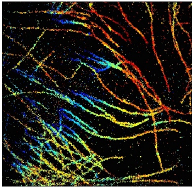
Mathematics of information



super-resolution microscopy

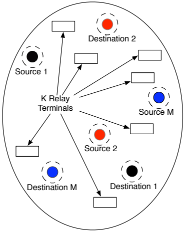
radar imaging

Mathematics of information

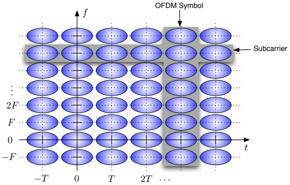


super-resolution microscopy

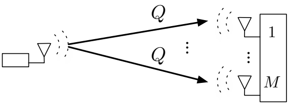
radar imaging



large wireless
networks

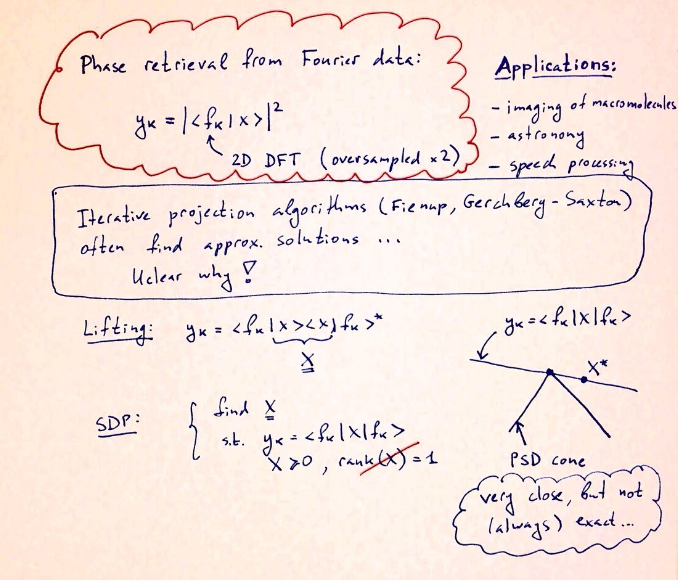


communication
under
channel uncertainty



systems
with
multiple receive antennas

Related open problems



Thank you